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# Gravitational waves from compact binaries in post-Newtonian accurate hyperbolic orbits

Gihyuk Cho,<sup>1,\*</sup> Achamveedu Gopakumar,<sup>2</sup> Maria Haney,<sup>3</sup> and Hyung Mok Lee<sup>4</sup>

<sup>1</sup>*Department of Physics and Astronomy, Seoul National University, Seoul 151-742, Korea*

<sup>2</sup>*Department of Astronomy and Astrophysics, Tata Institute of Fundamental Research, Mumbai 400005, India*

<sup>3</sup>*Physik-Institut, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland*

<sup>4</sup>*Korea Astronomy and Space Science Institute, Daejeon 34055, Korea*



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We derive from first principles a third post-Newtonian (3PN) accurate Keplerian-type parametric solution to describe PN accurate dynamics of nonspinning compact binaries in hyperbolic orbits. Orbital elements and functions of the parametric solution are obtained in terms of the conserved orbital energy and angular momentum in both Arnowitt-Deser-Misner-type and modified harmonic coordinates. Elegant checks are provided that include a modified analytic continuation prescription to obtain our independent hyperbolic parametric solution from its eccentric version. A prescription to model gravitational wave polarization states for hyperbolic compact binaries experiencing 3.5PN accurate orbital motion is presented that employs our 3PN accurate parametric solution.

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## I. INTRODUCTION

Interesting astrophysical scenarios involving strong gravitational fields usually require accurate and efficient ways of describing orbital dynamics of compact binaries. These scenarios include gravitational wave (GW) events, observed by the advanced LIGO-Virgo interferometers [1], labeled GW150914, GW151226, GW170104, GW170608, GW170814, and GW170817 [2–7]. The first five are associated with the coalescence of black hole (BH) binaries while GW170817 involved a merging neutron star binary. The other strong field scenarios involving compact binaries include radio observations of relativistic binary pulsars like PSR 1913 + 16 and PSR J0737-3039 [8,9] and optical observations of Blazar OJ287, powered by a massive BH binary central engine [10].

Orbital dynamics of compact binaries spiraling in due to the emission of GWs can be accurately described by the PN approximation to general relativity [11]. In this approximation, orbital dynamics of nonspinning compact binaries is provided as corrections to Newtonian equations of motion in powers of  $(v/c)^2 \sim Gm/(c^2 r)$ , where  $v$ ,  $m$ , and  $r$  are the velocity, total mass and relative separation of the binary, respectively. At present, conservative orbital dynamics of compact binaries have been computed to the fourth PN order which provides  $(v/c)^8$  accurate general relativity based corrections to Newtonian description (see for example Refs. [12–16] and their many references for a glimpse of

this herculean effort from various approaches). Interestingly, it is possible to obtain a Keplerian-type parametric solution to PN accurate orbital dynamics of compact binaries in noncircular orbits. This was demonstrated by Damour and Deruelle for 1PN accurate compact binary orbital dynamics, relevant for both eccentric and hyperbolic orbits [17]. They introduced three eccentricities so that the parametrization looks “Keplerian” even at 1PN order. These computations were extended to 2PN and 3PN orders by Schäfer and his collaborators which led to the generalized quasi-Keplerian parametric solution for compact binaries in precessing eccentric orbits [18–20]. This solution plays an important role in the ongoing efforts to model GWs from merging BH binaries in eccentric orbits [21,22]. This is due to the use of a certain GW phasing formalism, developed in Refs. [23,24], for describing the inspiral part of eccentric binary coalescence. This formalism employs a Keplerian-type parametric solution to model the orbital and periastron precession timescale variations present in the two GW polarization states  $h_+(t)$  and  $h_\times(t)$ . These features are crucial to obtain  $h_+(t)$  and  $h_\times(t)$  from compact binaries inspiraling along PN accurate eccentric orbits in an accurate and efficient manner [25]. Additionally, high precision radio observations of binary pulsars employ an accurate relativistic “timing formula” [26,27] which requires a 1PN accurate Keplerian-type parametric solution for compact binaries moving in precessing eccentric orbits [17]. This timing formula is crucial to test both the predictions of general relativity and the viability of alternate theories of gravity in strong field situations present in our galaxy [28].

\*whrlsos@snu.ac.kr

In this paper, we derive from first principles a parametric solution to 3PN accurate conservative orbital dynamics of compact binaries moving in hyperbolic orbits. This parametric solution is given both in Arnowitt-Deser-Misner-type (ADM) and modified harmonic (MH) coordinates. The reason that we adopt both gauges is that ADM is useful for comparing with numerical data from numerical relativity (NR) simulations which make use of ADM formalism, and MH is proper for constructing GW waveforms. The associated orbital elements and functions are provided as PN accurate functions of the conserved orbital energy, angular momentum and the symmetric mass ratio. The correctness of our solutions is verified by comparing 3PN accurate expressions for the radial and angular velocities arising from our solutions with their counterparts, computed directly from the orbital dynamics. Additionally, we develop a modified analytic continuation prescription to obtain our 3PN accurate Keplerian-type parametric solution for hyperbolic orbits from its eccentric versions, available in Ref. [20]. This is a desirable feature as we are essentially providing an additional test on the correctness of lengthy expressions present in Ref. [20] which are, as noted earlier, required to construct templates for eccentric inspirals. We also obtain temporally evolving GW polarization states for compact binaries in 3.5PN accurate hyperbolic orbits. This is achieved by allowing orbital elements and functions of our 3PN accurate Keplerian-type parametric solution to vary due to 1PN accurate radiation reaction effects, relevant for hyperbolic orbits [24,29]. Our efforts are motivated by the observation that compact binaries in unbound orbits are plausible GW sources for both the ground and space based GW observatories. It turns out that such rare events are expected to occur in globular clusters and galactic nuclear clusters or plausibly in dense clusters of primordial black holes [30,31]. Moreover, hyperbolic encounters may create bound binaries having very high eccentricities [32,33]. It was argued that plausible detection rates for such eccentric binaries may become comparable to that for isolated compact binary coalescences [34]. Interestingly, such hyperbolic GW events involving neutron stars may even be accompanied by electromagnetic flares [35]. The present effort should provide accurate gravitational waveforms for hyperbolic passages that can be adopted into the LSC Algorithm Library Suite of the LIGO Scientific Collaboration.

The paper is organized in the following way. In Sec. II we first present the derivation of a 1PN accurate Keplerian-type parametric solution for hyperbolic orbits from first principles. This is followed by detailing our approach to extend it to 3PN order and ways to check the correctness of our solutions in two different gauges. We present an accurate and efficient way to obtain temporally evolving GW polarization states for nonspinning compact binaries moving in 3.5PN accurate hyperbolic orbits in Sec. III. A brief summary and possible extensions are listed in Sec. IV.

## II. DERIVATION OF KEPLERIAN-TYPE PARAMETRIC SOLUTION FOR COMPACT BINARIES IN HYPERBOLIC ORBITS

We first provide a detailed derivation of a 1PN accurate Keplerian-type parametric solution for compact binaries in hyperbolic orbits. This procedure explicitly demonstrates why one requires three eccentricity parameters to obtain the desired solution, which was previously computed by employing certain analytic continuation arguments in Ref. [17]. The 3PN extension of Sec. II A is detailed in Sec. II B.

### A. 1PN accurate quasi-Keplerian parametrization for hyperbolic orbits

We begin by displaying 1PN accurate expressions for the radial and angular motion,

$$\left(\frac{dr}{dt}\right)^2 = a_0 + a_1 s + a_2 s^2 + a_3 s^3, \quad (2.1a)$$

$$\frac{d\phi}{dt} = d_0 s^2 + d_1 s^3, \quad (2.1b)$$

where both radial and temporal variables are scaled by  $Gm$  [17]. This allows us to introduce a variable  $s = 1/r$ , where  $r = |\mathbf{R}|/(Gm)$  and  $\mathbf{R}$  is the relative separation vector such that  $\mathbf{R} = R(\cos \phi, \sin \phi, 0)$ . The constant coefficients,  $a_0, a_1, a_2, a_3, d_0$  and  $d_1$  are given in terms of certain conserved orbital energy ( $\tilde{E}$ ) and angular momentum ( $\tilde{J}$ ) as

$$a_0 = \frac{2\tilde{E}}{\mu} + \frac{1}{c^2} \frac{(2\tilde{E})^2 - 3 + 9\eta}{\mu^2}, \quad (2.2a)$$

$$a_1 = 2 - \frac{1}{c^2} \frac{2\tilde{E}}{\mu} (6 - 7\eta), \quad (2.2b)$$

$$a_2 = -\frac{\tilde{J}^2}{G^2 m^2 \mu^2} - \frac{1}{c^2} \left[ \frac{2\tilde{E}\tilde{J}^2}{G^2 m^2 \mu^3} (3\eta - 1) + (-10 + 5\eta) \right], \quad (2.2c)$$

$$a_3 = \frac{1}{c^2} \frac{\tilde{J}^2}{G^2 m^2 \mu^2} (8 - 3\eta), \quad (2.2d)$$

$$d_0 = \frac{\tilde{J}}{Gm\mu} + \frac{1}{c^2} \frac{2\tilde{E}\tilde{J}}{G^2 m^2 \mu^3} \frac{-1 + 3\eta}{2}, \quad (2.2e)$$

$$d_1 = \frac{1}{c^2} \frac{\tilde{J}}{Gm\mu} (-4 + 2\eta), \quad (2.2f)$$

where  $\mu$  and  $\eta$  denote the usual reduced mass and symmetric mass ratio. It should be obvious that these coefficients take simpler forms in terms of certain reduced

orbital energy and angular momentum variables, defined as  $E = \tilde{E}/\mu$ ,  $h = \tilde{J}/(Gm\mu)$ . Additionally, we are considering unbound hyperbolic orbits; therefore  $E > 0$ .

Influenced by Ref. [17], we tackle the radial motion by introducing a certain *conchoidal* transformation

$$r = \bar{r} + \frac{a_3}{2a'_2} \quad (2.3)$$

where  $a'_2 = -h^2$  so that  $a_3/2a'_2 \sim O(\frac{1}{c^2})$  and  $\lim_{c^{-1} \rightarrow 0} a_2$  gives  $a'_2$ . It is fairly straightforward to recast the above radial equation in terms of  $\bar{r}$  as

$$\begin{aligned} \left(\frac{d\bar{r}}{dt}\right)^2 &= a_0 + \frac{a_1}{\bar{r}} + \frac{a_2}{\bar{r}^2} - \frac{a_1 a_3}{2a'_2 \bar{r}^2} + O\left(\frac{1}{c^4}\right), \\ &= a_0 + \frac{a_1}{\bar{r}} + \frac{\bar{a}_2}{\bar{r}^2} + O\left(\frac{1}{c^4}\right), \end{aligned} \quad (2.4)$$

where  $\bar{a}_2 = a_2 - \frac{a_1 a_3}{2a'_2}$ , while consistently neglecting terms  $O(\frac{1}{c^4})$ . To solve the above equation, we introduce an angular parameter  $u$  such that

$$\frac{du}{dt} = \frac{1}{a_4 \bar{r}} > 0; \quad a_4 > 0. \quad (2.5)$$

This leads to

$$\left(\frac{d\bar{r}}{du}\right)^2 = a_4^2(a_0 \bar{r}^2 + a_1 \bar{r} + \bar{a}_2). \quad (2.6)$$

Clearly, we require  $(a_0 \bar{r}^2 + a_1 \bar{r} + \bar{a}_2) > 0$ , and this allows us to write

$$\begin{aligned} \pm a_4 du &= \frac{d\bar{r}}{\sqrt{a_0 \bar{r}^2 + a_1 \bar{r} + \bar{a}_2}} \\ &= \frac{d\bar{r}}{\sqrt{\bar{a}_2 - \frac{a_1^2}{4a_0} + a_0(\bar{r} + \frac{a_1}{2a_0})^2}}. \end{aligned} \quad (2.7)$$

For hyperbolic motion, we observe that  $(\bar{a}_2 - a_1^2/(4a_0))$  is indeed negative. Therefore, we rewrite this equation as

$$\pm a_4 \left( \sqrt{-\bar{a}_2 + \frac{a_1^2}{4a_0}} \right) du = \frac{d\bar{r}}{\sqrt{-1 + \frac{4a_0^2}{a_1^2 - 4\bar{a}_2 a_0} (\bar{r} + \frac{a_1}{2a_0})^2}}. \quad (2.8)$$

We now introduce  $u'$  such that  $\cosh u' = \sqrt{\frac{4a_0^2}{a_1^2 - 4\bar{a}_2 a_0}} (\bar{r} + \frac{a_1}{2a_0})$  which allows us to simplify the above equation as

$$\pm a_4 \sqrt{a_0} du = du'. \quad (2.9)$$

We let  $a_4 = \frac{1}{\sqrt{a_0}}$  to ensure that  $\cosh u' = \cosh(\pm u) = \cosh u$ . This leads to the following equations for  $\bar{r}$  as well as  $r$ :

$$\bar{r} = -\frac{a_1}{2a_0} + \sqrt{\frac{a_1^2 - 4\bar{a}_2 a_0}{4a_0^2}} \cosh u, \quad (2.10a)$$

$$\begin{aligned} r &= -\frac{a_1}{2a_0} + \frac{a_3}{2a'_2} + \sqrt{\frac{a_1^2 - 4\bar{a}_2 a_0}{a_0^2}} \cosh u, \\ &= \left(\frac{a_1}{2a_0} - \frac{a_3}{2a'_2}\right) \left[-1 + \left(\frac{a_1}{2a_0} - \frac{a_3}{2a'_2}\right)^{-1} \times \sqrt{\frac{a_1^2 - 4\bar{a}_2 a_0}{4a_0^2}} \cosh u\right]. \end{aligned} \quad (2.10b)$$

We now identify  $(\frac{a_1}{2a_0} - \frac{a_3}{2a'_2})$  with  $a_r$  and the coefficient of  $\cosh u$  with  $e_r$ . The 1PN accurate expression for  $e_r$  is therefore given by

$$\begin{aligned} e_r &= \left(\frac{a_1}{2a_0} - \frac{a_3}{2a'_2}\right)^{-1} \sqrt{\frac{a_1^2 - 4\bar{a}_2 a_0}{4a_0^2}} \\ &= \left(1 + \frac{a_0 a_3}{a_1 a'_2}\right) \sqrt{1 - \frac{4\bar{a}_2 a_0}{a_1^2}} + O\left(\frac{1}{c^4}\right). \end{aligned} \quad (2.11)$$

Invoking Eqs. (2.2), the parametric equation for  $r$  may be summarized as

$$r = a_r(e_r \cosh u - 1), \quad (2.12a)$$

$$a_r = \frac{1}{2E} + \frac{1}{c^2} \left(\frac{7}{4} - \frac{\eta}{4}\right), \quad (2.12b)$$

$$\begin{aligned} e_r^2 &= 1 + 2Eh^2 \\ &+ \frac{1}{c^2} (2E) \left[-44 + 19\eta + 2Eh^2 \left(\frac{1}{2} - \frac{\eta}{2}\right)\right]. \end{aligned} \quad (2.12c)$$

We have verified that the expression for  $a_r$  is identical to Eq. (7.4) in Ref. [17], obtained by invoking the argument of analytic continuation.

To obtain the 1PN accurate Kepler equation for hyperbolic orbits, we turn to Eq. (2.5) for the angular variable  $u$  and integrate it. This leads to

$$\begin{aligned} \sqrt{a_0}(t - t_0) &= \int \bar{r} du, \\ &= \int du \left( -\frac{a_1}{2a_0} + \sqrt{\frac{a_1^2 - 4\bar{a}_2 a_0}{4a_0^2}} \cosh u \right), \\ &= \left( -\frac{a_1}{2a_0} u + \sqrt{\frac{a_1^2 - 4\bar{a}_2 a_0}{4a_0^2}} \sinh u \right). \end{aligned} \quad (2.13)$$

It is straightforward to rewrite the above equation in its more familiar form

$$n(t - t_0) = e_t \sinh u - u, \quad \text{where} \quad (2.14a)$$

$$n = \frac{2a_0^{\frac{3}{2}}}{a_1}, \quad (2.14b)$$

$$e_t = \sqrt{1 - \frac{4\tilde{a}_2 a_0}{a_1^2}}. \quad (2.14c)$$

Using Eqs. (2.2), we can express the orbital elements  $n$  and  $e_t^2$  in terms of the conserved  $E$ ,  $h$  and  $\eta$  as

$$n = (2E)^{\frac{3}{2}} + \frac{1}{c^2} (2E)^{\frac{5}{2}} \frac{15 - \eta}{8}, \quad (2.15a)$$

$$e_t^2 = 1 + 2Eh^2 + \frac{1}{c^2} (2E) \left[ -18 + 8\eta + 2Eh^2 \left( \frac{17}{4} - \eta \frac{7}{4} \right) \right]. \quad (2.15b)$$

The above expressions are also identical to those given in Ref. [17].

We are now in a position to tackle the angular motion. Influenced by Ref. [17], we employ another conchoidal transformation

$$\tilde{r} = r - \frac{d_1}{2d_0}. \quad (2.16)$$

Using our expression for  $r$ , we obtain

$$\tilde{r} = \tilde{a}(\tilde{e} \cosh u - 1), \quad (2.17)$$

where  $\tilde{a} = a_r - \frac{d_1}{2d_0}$  and  $\tilde{e} = \frac{a_r e_r}{\tilde{a}}$ . In terms of  $\tilde{r}$ , the 1PN accurate equation for the angular motion, given in Eqs. (2.1), takes the simpler Newtonian form:

$$\frac{d\phi}{dt} = \frac{d_0}{\tilde{r}^2}. \quad (2.18)$$

With the help of our 1PN accurate Kepler equation, this leads to

$$\frac{d\phi}{du} = \frac{d_0}{n\tilde{a}^2} \frac{(e_t \cosh u - 1)}{(\tilde{e} \cosh u - 1)^2}. \quad (2.19)$$

We introduce  $e_\phi = 2\tilde{e} - e_t$  which allows us to simplify  $(e_t \cosh u - 1)/((\tilde{e} \cosh u - 1)^2)$  as  $1/(e_\phi \cosh u - 1)$ , modulo the neglected  $\mathcal{O}(1/c^4)$  terms. Integrating the resulting expression for  $d\phi/du = d_0/(n\tilde{a}^2(e_\phi \cosh u - 1))$  gives us

$$\begin{aligned} \phi - \phi_0 &= \frac{d_0}{n\tilde{a}^2} \int \frac{du}{e_\phi \cosh u - 1} \\ &= \frac{d_0}{n\tilde{a}^2} \int \frac{du}{(e_\phi - 1) \cosh^2 \frac{u}{2} + (e_\phi + 1) \sinh^2 \frac{u}{2}} \\ &= \frac{d_0}{n(\sqrt{e_\phi^2 - 1})\tilde{a}^2} \int \frac{du \sqrt{\frac{e_\phi + 1}{e_\phi - 1}} \frac{1}{\cosh^2 \frac{u}{2}}}{1 + \frac{e_\phi + 1}{e_\phi - 1} \tanh^2 \frac{u}{2}} \\ &= \frac{d_0}{n\tilde{a}^2 \sqrt{e_\phi^2 - 1}} 2 \arctan \left( \sqrt{\frac{e_\phi + 1}{e_\phi - 1}} \tanh \frac{u}{2} \right). \end{aligned} \quad (2.20)$$

We now introduce  $K$  such that

$$\phi - \phi_0 = K \times 2 \arctan \left( \sqrt{\frac{e_\phi + 1}{e_\phi - 1}} \tanh \frac{u}{2} \right), \quad \text{where} \quad (2.21a)$$

$$K = \frac{d_0}{n\tilde{a}^2 \sqrt{e_\phi^2 - 1}}. \quad (2.21b)$$

It is straightforward to express the orbital elements  $K$  and  $e_\phi$  in terms of the conserved quantities like  $E$  and  $h$ , and we have

$$K = 1 + \frac{1}{c^2} \frac{34 - 15\eta + (2Eh^2)(-8 + 3\eta)}{2h^2}, \quad (2.22a)$$

$$\begin{aligned} e_\phi^2 &= 1 + 2Eh^2 \\ &+ \frac{1}{c^2} (2E) \left[ -34 + 15\eta + 2Eh^2 \left( \frac{-47}{4} + \frac{21}{4}\eta \right) \right]. \end{aligned} \quad (2.22b)$$

We have verified that Eqs. (2.12), (2.14), (2.15), (2.21), and (2.22) indeed are identical to their counterparts in Ref. [17] that have been obtained by invoking the arguments of analytic continuation. These arguments establish that the parametric elliptic solution (i.e., in the case of  $E < 0$ ) is well defined and analytic in  $E$  and  $u$  even in the domain where  $E > 0$  and  $u \rightarrow iu$  is purely imaginary. The solution for the 1PN accurate hyperbolic motion obtained in Ref. [17] through analytic continuation to  $E > 0$  and purely imaginary  $u$  has been found to be the same as Eqs. (2.12), (2.14), (2.15), (2.21), and (2.22). Note that the three distinct eccentricity parameters  $e_t$ ,  $e_\phi$ ,  $e_r$  ensure that the 1PN accurate parametric solution looks quasi-Keplerian. The presence of  $K$  can modify the trajectories of PN accurate hyperbolic orbits with respect to their Newtonian counterparts, as we will demonstrate later. In the next subsection, we extend these calculations to 3PN order.



### B. 3PN accurate hyperbolic generalized quasi-Keplerian parametrization for compact binaries

The plan is to derive from first principles a 3PN accurate Keplerian-type parametric solution for compact binaries in hyperbolic orbits. We are attempting the 3PN extension of Sec. II A, as it is not straightforward to obtain a hyperbolic version of the 3PN accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits, detailed in Ref. [20], simply by invoking the analytic continuation arguments of Ref. [17]. The main difficulty with analytic continuation is due to the structure of the 3PN accurate (eccentric) Kepler equation, given by Eq. (19b) in Ref. [20], which reads

$$l \equiv n'(t - t_0) = u' - e_l \sin u' + \left( \frac{g_{4l'}}{c^4} + \frac{g_{6l'}}{c^6} \right) (v' - u') + \left( \frac{f_{4l'}}{c^4} + \frac{f_{6l'}}{c^6} \right) \sin v' + \frac{i_{6l'}}{c^6} \sin 2v' + \frac{h_{6l'}}{c^6} \sin 3v', \quad (2.23)$$

where  $n'$ ,  $u'$  and  $v'$  stand for the mean motion, eccentric and true anomalies of an eccentric orbit, and where the orbital functions  $g_{4l'}$ ,  $g_{6l'}$ ,  $f_{4l'}$ ,  $f_{6l'}$ ,  $i_{6l'}$ ,  $h_{6l'}$  are PN accurate functions of the conserved energy, angular momentum and the symmetric mass ratio  $\eta$ . It is customary to employ the following exact expression for  $(v' - u')$ , derived in Ref. [24]:

$$v' - u' = 2 \tan^{-1} \left( \frac{\beta'_{\phi} \sin u'}{1 - \beta'_{\phi} \cos u'} \right), \quad (2.24)$$

where  $\beta'_{\phi} = (1 - \sqrt{1 - e_{\phi}^2})/e_{\phi}$ . A close inspection reveals that it is certainly problematic to apply the usual analytic continuation arguments of [17], namely, to let  $u' \rightarrow \iota v$  and allow  $\sqrt{-E} \rightarrow \iota \sqrt{E}$  to obtain the hyperbolic version of an exact expression for  $v' - u'$ . Additionally, the presence of  $(-2Eh^2)^{1/2}$  and its multiples in the explicit expressions for  $n'$ ,  $g_{4l'}$  and  $g_{6l'}$ , as given by Eqs. (20) of Ref. [20], introduces further complications while trying to achieve hyperbolic versions of these expressions.

These considerations prompted us to obtain hyperbolic versions of Eqs. (19)–(21) of Ref. [20] (describing the radial and angular motion in an eccentric binary, as well as the Kepler equation) with the help of *ab initio* computations. It turned out that these detailed computations enabled us to devise a modified version of the standard analytic continuation arguments, in order to extract hyperbolic counterparts of the expressions in Ref. [20]. This allowed us to check the correctness of our computations and to confirm the validity of Ref. [20]. An additional way of checking the lengthy expressions in Ref. [20] is highly desirable, as this work is usually invoked for the GW

phasing of compact binaries inspiraling along eccentric orbits.

We begin by tackling the hyperbolic radial part of the 3PN accurate Keplerian-type parametric solution. The input for our calculation is the following 3PN accurate expression for  $\dot{r}^2$ , symbolically written as

$$\dot{r}^2 \equiv \frac{1}{s^4} \left( \frac{ds}{dt} \right)^2 = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7, \quad (2.25)$$

where explicit functional forms of the coefficients  $a_i$  are given by Eqs. (A1) and (A3) of Ref. [20], for the ADM-type and modified harmonic gauges, respectively. We observe that in the Newtonian limit the right-hand side of Eq. (2.25) is a second order polynomial in  $s$  and therefore admits two roots. It is straightforward to obtain 3PN accurate versions of these two real-valued roots, even in the case of hyperbolic orbits. Subsequently, we factorize the 3PN accurate expression for  $\dot{r}^2$  with the help of the two roots  $s_+$  and  $s_-$ . This leads to

$$(t - t_0) = \int \frac{ds(b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4 + b_5 s^5)}{s^2 \sqrt{(s_+ - s)(s - s_-)}}, \quad (2.26)$$

where we used the parametric equation  $r = a_r(e_r \cosh u - 1)$ . The explicit functional forms for the coefficients  $b_i$  may be found in Eqs. (A2) and (A4) of Ref. [20]. Note the factorization of the denominator and how it differs from Eq. (9) of Ref. [20]. This is because for hyperbolic orbits we find the roots  $s_+ > 0$  and  $s_- < 0$ , which allows us to introduce  $r = a_r(e_r \cosh u - 1)$ . The above integral leads to

$$t - t_0 = c'_0(e_r \sinh u - u) + c'_1 u + \frac{c'_2 \nu'}{\sqrt{e_r^2 - 1}} + \frac{c'_3}{(e_r^2 - 1)^{3/2}} (\nu' + e_r \sin \nu') + \frac{c'_4}{(e_r^2 - 1)^{5/2}} \left( \frac{e_r^2 + 2}{2} \nu' + 2e_r \sin \nu' + \frac{e_r^2}{4} \sin 2\nu' \right) + \frac{c'_5}{(e_r^2 - 1)^{7/2}} \left[ \left( 1 + \frac{3e_r^2}{2} \right) \nu' + \left( 3e_r + \frac{3}{4} e_r^3 \right) \sin \nu' + \frac{3e_r^2}{4} \sin 2\nu' + \frac{e_r^3}{12} \sin 3\nu' \right], \quad (2.27)$$

where  $c'_i = b_i / (a_r^{i-1} \sqrt{-s_+ s_-})$  and  $\nu' = 2 \frac{1}{\sqrt{e_r^2 - 1}} \arctan \times \left( \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \frac{u}{2} \right)$ . The above equation can be rewritten as

$$t - t_0 = c_0 \sinh u - c_1 u + c_2 \nu' + c_3 \sin \nu' + c_4 \sin 2\nu' + c_5 \sin 3\nu', \quad (2.28)$$

with explicit relations between the coefficients  $c_i$ s and  $c'_i$ s given in Appendix C.

It is straightforward to deduce that the coefficient  $c_3$  of  $\sin \nu'$  in Eq. (2.28) begins at 1PN order. Therefore, the above result deviates from our 1PN accurate Keplerian-type parametric solution, derived in the previous section. It turns out that a suitable change of the  $\nu'$  variable can remedy this undesirable feature, which will be addressed later.

We turn our attention to the angular motion. The relevant ingredient of the calculation is  $d\phi/ds = \dot{\phi}/\dot{s}$ , which may be symbolically written as

$$\frac{d\phi}{ds} = -\frac{d_0 + d_1 s + d_2 s^2 + d_3 s^3 + d_4 s^4 + d_5 s^5}{\sqrt{(s_+ - s)(s - s_-)}}, \quad (2.29)$$

where the coefficients  $d_i$  are listed in Eqs. (A2) and (A4) in Ref. [20] (there denoted as  $B_i$ ), for ADM-type and modified harmonic gauges, respectively. This leads to

$$\phi - \phi_0 = \int du \left( \frac{e'_0}{r} + \frac{e'_1}{r^2} + \frac{e'_2}{r^3} + \frac{e'_3}{r^4} + \frac{e'_4}{r^5} + \frac{e'_5}{r^6} \right), \quad (2.30)$$

where  $e'_i = d_i/(a_r^{i+1} \sqrt{-s_+ s_-})$ . The above expression can be integrated to obtain

$$\phi - \phi_0 = e_0 \nu' + e_1 \sin \nu' + e_2 \sin 2\nu' + e_3 \sin 3\nu' + e_4 \sin 4\nu' + e_5 \sin 5\nu'. \quad (2.31)$$

As expected, the coefficients  $e_i$  are certain PN accurate expressions and are given as functions of  $e'_i$  in Appendix C. We observe that the coefficient of  $\sin \nu'$  in Eq. (2.31), namely  $e_1$ , begins at 1PN order. Therefore, the above functional form for the angular motion  $\phi - \phi_0$  also deviates from our 1PN accurate angular solution, given by Eqs. (2.21).

It is possible to correct this undesirable feature by introducing a certain PN accurate true anomaly  $\nu = 2 \arctan[(\frac{e_\phi + 1}{e_\phi - 1})^{1/2} \tanh \frac{u}{2}]$ , defined with the help of the angular eccentricity  $e_\phi$ . This eccentricity parameter deviates from  $e_r$  at PN orders by yet to be computed PN corrections. It is easy to obtain the following 3PN accurate expression for  $\nu'$  in terms of  $\nu$ ,

$$\nu' = \nu + \left( f' - \frac{f'^2}{2} + \frac{f'^3}{4} \right) \sin \nu + \left( \frac{f'^2}{4} - \frac{f'^3}{4} \right) \sin 2\nu + \frac{f'^3}{12} \sin 3\nu, \quad (2.32)$$

where  $f'$  should provide PN corrections connecting  $e_\phi$  and  $e_r$ . We invoke the above relation in our  $\phi - \phi_0$ , given by

Eq. (2.31), and demand that there be no  $\sin \nu'$  terms to 3PN order. The resulting 3PN accurate parametric solution for the angular motion indeed reproduces Eq. (2.31) when restricted to 1PN order. This procedure uniquely provides the PN corrections that connect  $e_\phi$  to  $e_r$ , and the resulting final parametrization for the angular motion reads

$$\begin{aligned} \frac{2\pi}{\Phi}(\phi - \phi_0) = \nu + \left( \frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6} \right) \sin 2\nu + \left( \frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6} \right) \\ \times \sin 3\nu + \frac{h_{6\phi}}{c^6} \sin 4\nu + \frac{i_{6\phi}}{c^6} \sin 5\nu. \end{aligned} \quad (2.33)$$

We are now in a position to reparametrize our 3PN accurate expression for  $t - t_0$ , given by Eq. (2.28), in terms of  $\nu$  with a procedure similar to the one outlined above. This also ensures that we recover our Keplerian-type parametric expression for  $l(u)$  at 1PN order. The improved expression for the 3PN accurate Kepler equation reads

$$\begin{aligned} \frac{2\pi}{P}(t - t_0) = e_t \sinh u - u + \left( \frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6} \right) \nu \\ + \left( \frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6} \right) \sin \nu + \frac{h_{6t}}{c^6} \sin 2\nu + \frac{i_{6t}}{c^6} \sin 3\nu. \end{aligned} \quad (2.34)$$

We observe that the transformation from  $\nu'$  to  $\nu$  ensures that the coefficients of  $\nu$  terms appear only at the 2PN and 3PN orders.

Collecting various results, we display in full the third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in hyperbolic orbits as

$$r = a_r(e_r \cosh u - 1), \quad (2.35a)$$

$$\begin{aligned} \frac{2\pi}{P}(t - t_0) = e_t \sinh u - u + \left( \frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6} \right) \nu + \left( \frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6} \right) \sin \nu \\ + \frac{h_{6t}}{c^6} \sin 2\nu + \frac{i_{6t}}{c^6} \sin 3\nu, \end{aligned} \quad (2.35b)$$

$$\begin{aligned} \frac{2\pi}{\Phi}(\phi - \phi_0) = \nu + \left( \frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6} \right) \sin 2\nu + \left( \frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6} \right) \sin 3\nu \\ + \frac{h_{6\phi}}{c^6} \sin 4\nu + \frac{i_{6\phi}}{c^6} \sin 5\nu, \end{aligned} \quad (2.35c)$$

where  $\nu = 2 \tan^{-1}[(\frac{e_\phi + 1}{e_\phi - 1})^{1/2} \tanh \frac{u}{2}]$ . Note that the 3PN accurate expressions for the orbital elements  $a_r$ ,  $e_r^2$ ,  $P = 2\pi/n$ ,  $e_t^2$ ,  $\Phi$ , and  $e_\phi^2$  and the orbital functions  $g_{4t}$ ,  $g_{6t}$ ,  $f_{4t}$ ,  $f_{6t}$ ,  $i_{6t}$ ,  $h_{6t}$ ,  $f_{4\phi}$ ,  $f_{6\phi}$ ,  $g_{4\phi}$ ,  $g_{6\phi}$ ,  $i_{6\phi}$ , and  $h_{6\phi}$  are functions of  $E$ ,  $h$  and  $\eta$ . Their 3PN accurate expressions in the modified harmonic coordinates arise from Eqs. (A3) and (A4) of Ref. [20] and are given by

$$\begin{aligned}
a_r = & \frac{1}{(2E)} \left\{ 1 + \frac{(2E)}{4c^2} (7 - \eta) + \frac{(2E)^2}{16c^4} \left[ (1 + \eta^2) + \frac{1}{(2Eh^2)} (64 - 112\eta) \right] \right. \\
& + \frac{(2E)^3}{192c^6} \left[ -3 + 3\eta - 3\eta^3 + \frac{1}{(2Eh^2)} \left( 768 + \left( 123\pi^2 - \frac{215408}{35} \right) \eta + 1344\eta^2 \right) \right. \\
& \left. \left. + \frac{1}{(2Eh^2)^2} \left( 6144 + \left( -\frac{704096}{35} + 492\pi^2 \right) \eta + 1728\eta^2 \right) \right] \right\}, \quad (2.36a)
\end{aligned}$$

$$\begin{aligned}
e_r^2 = & 1 + 2Eh^2 + \frac{(2E)}{4c^2} \{-24 + 4\eta + 5(-3 + \eta)(2Eh^2)\} \\
& + \frac{(2E)^2}{8c^4} \left\{ 60 + 148\eta + 2\eta^2 + (80 - 45\eta + 4\eta^2)(2Eh^2) + \frac{8}{(2Eh^2)} (-16 + 28\eta) \right\} \\
& + \frac{(2E)^3}{6720c^6} \left\{ 2(1680 - (90632 + 4305\pi^2)\eta + 33600\eta^2) + 4\eta^3 - \frac{80}{(2Eh^2)} (1008 + (-21130 + 861\pi^2)\eta + 2268\eta^2) \right. \\
& \left. - \frac{16}{(2Eh^2)^2} ((53760 + (-176024 + 4305\pi^2)\eta + 15120\eta^2)) \right\}, \quad (2.36b)
\end{aligned}$$

$$n = (2E)^{3/2} \left\{ 1 - \frac{(2E)}{8c^2} (-15 + \eta) + \frac{(2E)^2}{128c^4} [555 + 30\eta + 11\eta^2] + \frac{(2E)^3}{1024c^6} [653 + 111\eta + 7\eta^2 + 3\eta^3] \right\}, \quad (2.36c)$$

$$\begin{aligned}
e_t^2 = & 1 + 2Eh^2 + \frac{(2E)}{4c^2} \{8 - 8\eta + (17 - 7\eta)(2Eh^2)\} \\
& + \frac{(2E)^2}{8c^4} \left\{ 4(3 + 18\eta + 5\eta^2) + (2Eh^2)(112 - 47\eta + 16\eta^2) + \frac{16}{(2Eh^2)} (-4 + 7\eta) \right\} \\
& + \frac{(2E)^3}{840c^6} \left\{ -70(42 - 830\eta + 321\eta^2 + 30\eta^3) - \frac{525}{8} (2Eh^2)(-528 + 200\eta - 77\eta^2 + 24\eta^3) \right. \\
& \left. - \frac{3}{4(2Eh^2)} (73920 + (-260272 + 4305\pi^2)\eta + 61040\eta^2) - \frac{1}{(2Eh^2)^2} (53760 + (-176024 + 4305\pi^2)\eta + 15120\eta^2) \right\}, \quad (2.36d)
\end{aligned}$$

$$f_{4t} = \frac{3(2E)^2}{2} \left\{ \frac{5 - 2\eta}{\sqrt{(2Eh^2)}} \right\}, \quad (2.36e)$$

$$f_{6t} = \frac{(2E)^3}{192(2Eh^2)^{3/2}} \{ (10080 + 123\eta\pi^2 - 13952\eta + 1440\eta^2) + (2Eh^2)36(95 - 55\eta + 18\eta^2) \}, \quad (2.36f)$$

$$g_{4t} = -\frac{1}{8} \frac{(2E)^2}{\sqrt{(2Eh^2)}} \{ (-15 + \eta)\eta\sqrt{(1 + 2Eh^2)} \}, \quad (2.36g)$$

$$\begin{aligned}
g_{6t} = & \frac{(2E)^3}{2240(2Eh^2)^{3/2}\sqrt{1 + 2Eh^2}} \{ 35(2Eh^2)^2\eta(23\eta^2 - 175\eta + 297) + (2Eh^2)(22400 + (49321 - 1435\pi^2)\eta \\
& - 27300\eta^2 + 1225\eta^3) + 385\eta^3 - 20965\eta^2 + (-1435\pi^2 + 43651)\eta + 22400 \}, \quad (2.36h)
\end{aligned}$$

$$h_{6t} = \frac{(2E)^3}{16} \eta \left\{ \frac{(1 + 2Eh^2)}{(2Eh^2)^{3/2}} (116 - 49\eta + 3\eta^2) \right\}, \quad (2.36i)$$

$$i_{6t} = \frac{(2E)^3}{192} \eta^3 \left( \frac{1 + 2Eh^2}{2Eh^2} \right)^{3/2} (23 - 73\eta + 13\eta^2), \quad (2.36j)$$



$$\Phi = 2\pi \left\{ 1 + \frac{3}{c^2 h^2} + \frac{3(2E)^2}{4(2Eh^2)^2 c^4} [-35 + 10\eta + (2Eh^2)(-5 + 2\eta)] + \frac{(2E)^3}{128c^6(2Eh^2)^3} [36960 + (615\pi^2 - 40000)\eta + 1680\eta^2 + (2Eh^2)(10080 + 123\eta\pi^2 - 13952\eta + 1440\eta^2) + (2Eh^2)^2(120 - 120\eta + 96\eta^2)] \right\}, \quad (2.36k)$$

$$f_{4\phi} = \frac{(2E)^2 (1 + 2Eh^2)}{8 (2Eh^2)^2} (1 + 19\eta - 3\eta^2), \quad (2.36l)$$

$$f_{6\phi} = \frac{(2E)^3}{26880(2Eh^2)^3} \{67200 + (994704 - 30135\pi^2)\eta - 335160\eta^2 - 4200\eta^3 + 280(2Eh^2)^2(3 + 506\eta - 357\eta^2 + 36\eta^3) + (2Eh^2)(60480 + (991904 - 30135\eta^2)\eta - 428400\eta^2 + 8400\eta^3)\}, \quad (2.36m)$$

$$g_{4\phi} = \frac{(1 - 3\eta)(2E)^2}{32} \frac{\eta}{(2Eh^2)^2} (1 + 2Eh^2)^{3/2}, \quad (2.36n)$$

$$g_{6\phi} = \frac{(2E)^3 \sqrt{(1 + 2Eh^2)}}{768 (2Eh^2)^3} \eta \{36161 - 1435\pi^2 - 28525\eta + 525\eta^2 + 35(2Eh^2)^2(14 - 49\eta + 26\eta^2) + (2Eh^2)(35706 - 1435\pi^2 - 27510\eta + 1750\eta^2)\}, \quad (2.36o)$$

$$h_{6\phi} = \frac{(2E)^3 (1 + 2Eh^2)^2}{192 (2Eh^2)^3} \eta (82 - 57\eta + 15\eta^2), \quad (2.36p)$$

$$i_{6\phi} = \frac{(2E)^3}{256} \eta \frac{1 - 5\eta + 5\eta^2}{(2Eh^2)^3} (1 + 2Eh^2)^{5/2}, \quad (2.36q)$$

$$e_\phi^2 = 1 + 2Eh^2 + \frac{(2E)}{4c^2} \{-24 + (-15 + \eta)(2Eh^2)\} + \frac{(2E)^2}{16c^4(2Eh^2)} \{-416 + 91\eta + 15\eta^2 + 2(2Eh^2)(-20 + 17\eta + 9\eta^2) + (2Eh^2)^2(160 - 31\eta + 3\eta^2)\} - \frac{(2E)^3}{13440c^6(2Eh^2)^2} \{2956800 + (-5627206 + 81795\pi^2)\eta - 14490\eta^2 - 7350\eta^3 - (2Eh^2)^2(584640 + (17482 + 4305\pi^2)\eta + 7350\eta^2 - 8190\eta^3) + 420(2Eh^2)^3(744 - 248\eta + 31\eta^2 + 3\eta^3) + 14(2Eh^2)(36960 + 7(-48716 + 615\pi^2)\eta - 225\eta^2 + 150\eta^3)\}. \quad (2.36r)$$

Let us recall that both the radial and temporal coordinates are scaled by  $Gm$ , and that the expressions for  $a_r$  and  $n$  are therefore given by  $a_r = 1/(2E)$  and  $n = (2E)^{3/2}$  at the Newtonian order. The three eccentricities  $e_r$ ,  $e_t$  and  $e_\phi$ , which differ from each other from the first post-Newtonian order, are related by

$$e_t = e_r \left\{ 1 + \frac{(2E)}{2c^2} (8 - 3\eta) + \frac{(2E)^2}{c^4} \frac{1}{(2Eh^2)} [8 - 14\eta + (36 - 19\eta + 6\eta^2)(Eh^2)] + \frac{(2E)^3}{3360c^6} \frac{1}{(2Eh^2)^2} [-420(2Eh^2)^2(10\eta^3 - 34\eta^2 + 65\eta - 160) + Eh^2(105840\eta^2 + (4305\pi^2 - 354848)\eta + 87360) + 30240\eta^2 + (8610\pi^2 - 352048)\eta + 107520] \right\}, \quad (2.37a)$$

$$e_\phi = e_r \left\{ 1 - \frac{(2E)}{2c^2} \eta - \frac{(2E)^2}{32c^4} \frac{1}{(2Eh^2)} [160 + 357\eta - 15\eta^2 - \eta(-1 + 11\eta)(2Eh^2)] + \frac{(2E)^3}{8960c^6} \frac{1}{(2Eh^2)^2} [-70(2Eh^2)^2\eta(31\eta^2 - \eta - 1) + 5(2Eh^2)(-1050\eta^3 + 31304\eta^2 + (1435\pi^2 - 36546)\eta + 4928) + 2450\eta^3 + 166110\eta^2 + (18655\pi^2 - 1854)\eta - 412160] \right\}. \quad (2.37b)$$

These relations allow one to choose a specific eccentricity parameter to describe a PN accurate hyperbolic orbit.

Following the above detailed procedure, it is straightforward to obtain 3PN accurate expressions for the above listed quantities also in an ADM-type gauge. The 3PN accurate Keplerian-type parametric solution arises from Eqs. (A1) and (A2) of Ref. [20] and is structurally identical to Eqs. (A1). This is expected as Eqs. (A1)–(A4) of Ref. [20] are polynomials of the same degree though their coefficients are different. The ADM versions of Eqs. (2.36) are listed in Appendix A.

We are now in a position to explore the possibility of obtaining our 3PN accurate hyperbolic solution from its eccentric counterpart through analytic continuation. A close inspection of our results reveals that the 3PN accurate expression for the orbital element  $n$  in Eqs. (2.36) is structurally different from its eccentric counterpart, given by Eq. (25c) of Ref. [20]. Moreover, the structure of the relevant two Kepler equations is different [compare Eq. (19b) of Ref. [20] with our Eq. (2.35b)]. Therefore, it is reasonable to expect that additional arguments may be required to obtain *practically viable* analytic continuation arguments for extracting our main results from that of Ref. [20]. We begin from the eccentric Kepler equation, given by Eq. (24b) of Ref. [20], which may be written as

$$l = \frac{2\pi}{P_e}(t - t_0) = u' - e_t \sin u' + \left(\frac{g'_{4t}}{c^4} + \frac{g'_{6t}}{c^6}\right)(\nu' - u') \\ + \left(\frac{f'_{4t}}{c^4} + \frac{f'_{6t}}{c^6}\right) \sin \nu' + \frac{i'_{6t}}{c^6} \sin 2\nu' \\ + \frac{h'_{6t}}{c^6} \sin 3\nu', \quad (2.38)$$

where *primed* variables denote an eccentric binary and  $P_e$  stands for the 3PN accurate orbital period of an eccentric binary. The presence of the term  $\nu' - u' \equiv 2 \tan^{-1}(\beta'_\phi \sin u' / (1 - \beta'_\phi \cos u'))$  in the above Kepler equation, where  $\beta_\phi = (1 - \sqrt{1 - e_\phi^2})/e_\phi$ , leads to certain imaginary terms while adapting the usual argument of analytic continuation, namely  $u' \rightarrow i\nu$  and  $\sqrt{-E} \rightarrow i\sqrt{E}$ , to obtain its hyperbolic version [17].

At 1PN order, the above arguments ensure that the expression for  $P_e$  becomes a purely imaginary quantity, i.e.,  $iP_{\text{hyp}}$  and that  $u' - e_t \sin u'$  becomes  $i\nu - e_t \sin(i\nu)$ . This guarantees that  $(P_e)(u' - e_t \sin u')/(2\pi)$  leads to  $(P_{\text{hyp}})(e_t \sinh \nu - \nu)/(2\pi)$ . This observation influenced us to consider an expression for  $(t - t_0)$ , as given by Eq. (2.38), rather than  $l = n(t - t_0)$ , while invoking the usual arguments for analytic continuation (AAC). It is easy to show that the 3PN accurate eccentric expression for  $n$  gives a complex quantity rather than a purely imaginary one under the AAC. This is essentially due to the presence of  $(-2Eh^2)$  terms present in Eq. (25c) in Ref. [20]. Similar

arguments also apply to the terms  $u' - e_t \sin u' + (\frac{g'_{4t}}{c^4} + \frac{g'_{6t}}{c^6})(\nu' - u')$  in the 3PN accurate eccentric Kepler equation under the AAC. However, the product of  $P_e/2\pi$  and the above terms while substituting  $u' \rightarrow i\nu$  and  $\sqrt{-E} \rightarrow i\sqrt{E}$  becomes a real quantity and can be identified with  $(P/2\pi) \times [e_t \sinh u - u + (\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6})\nu]$ . Here,  $P/(2\pi)$  is the PN accurate inverse of  $n$ , given by our Eqs. (2.36). It should be noted that this procedure ensures that the complex quantities that we encountered while applying the AAC in Eq. (2.38) are now properly handled to obtain our 3PN accurate hyperbolic solution. Let us emphasize that we were able to formulate this reasoning mainly because of the availability of our 3PN accurate hyperbolic solution, obtained from our detailed *ab initio* computations. In other words, it is rather difficult to extract a 3PN accurate orbital element  $n$  for hyperbolic orbits from its eccentric version simply by invoking the arguments for analytic continuation of Ref. [17]. We require redefinitions of certain terms in the eccentric Kepler equation to obtain its hyperbolic version through analytic continuation. These redefinitions, however, can be worked out if the actual hyperbolic solution is available, computed from first principles as done in this paper. Finally, we observe that all other eccentric orbital elements and functions transition smoothly into their hyperbolic counterparts while employing the AAC. The extraction of our 3PN accurate hyperbolic solution from its eccentric counterpart, as noted earlier, provides an additional test for the correctness of the lengthy expressions present in Ref. [20].

We have also adapted for our purposes a consistency check which was devised in Ref. [20] to test the fidelity of the 3PN accurate eccentric parametrization and its PN accurate orbital elements and functions. The idea is to compute 3PN accurate expressions for  $i^2$  and  $\dot{\phi}^2$  using our parametric solution, via  $i^2 = (\frac{dr}{du} \frac{du}{dt})^2$  and  $\dot{\phi}^2 = (\frac{d\phi}{dv} \frac{dv}{du} \frac{du}{dt})^2$ . These lengthy 3PN accurate expressions are first obtained in terms of  $E$ ,  $h$ ,  $\eta$  and  $(e_r \cosh u - 1)$  and are later converted in terms of  $E$ ,  $h$ ,  $\eta$  and  $r$  while using our 3PN accurate expression for  $r = a_r(E, h, \eta)(e_r \cosh u - 1)$ . A detailed check is provided by comparing these parametric expressions for  $i^2$  and  $\dot{\phi}^2$  with those extracted from Eqs. (A3) and (A4) in Ref. [20]. Note that these equations arise from the 3PN accurate expressions for the orbital energy and angular momentum as evident by examining Eqs. (22) and (23) and the associated discussions in Ref. [20]. We have verified that the above two sets of 3PN accurate expressions for  $i^2$  and  $\dot{\phi}^2$  in terms  $E$ ,  $h$ ,  $\eta$  and  $r$  are identical to each other in the case of hyperbolic orbits. Let us emphasize that this check is very sensitive to the structure of the parametric solution and the explicit PN accurate expressions for the various orbital elements and functions. Therefore, the complete agreement to 3PN order between the parametric and Hamiltonian-based sets of  $i^2$  and  $\dot{\phi}^2$  expressions—along with our improved analytic

continuation relations—provide powerful checks on our 3PN accurate generalized quasi-Keplerian parametrization for compact binaries in hyperbolic orbits. Additionally, we have verified that our results are in agreement with Ref. [17] at 1PN order. In what follows, we apply our 3PN accurate Keplerian-type parametric solution to obtain time-domain gravitational waveforms for compact binaries in hyperbolic orbits while incorporating effects of GW emission.

### III. GW POLARIZATION STATES FOR COMPACT BINARIES IN 3.5PN ACCURATE HYPERBOLIC ORBITS

This section presents an efficient prescription to obtain temporally evolving GW polarization states for compact binaries moving in fully 3.5PN accurate hyperbolic orbits. Clearly, this requires us to prescribe a way of incorporating the dissipative effects of GW emission appearing at 2.5PN and 3.5PN orders into our 3PN accurate orbital dynamics. With the help of Refs. [23,24,29], this is pursued in steps which we will briefly outline below. We begin by considering the following expressions for the quadrupolar (or Newtonian) order GW polarization states,  $h_{+|Q}$  and  $h_{\times|Q}$ , for compact binaries in noncircular orbits, available in Ref. [29], which read

$$h_{+|Q} = -\frac{Gm\eta}{c^4 R} \quad (3.1a)$$

$$\times \{(1 + C_\theta^2)[(z + r^2\dot{\phi}^2 - \dot{r}^2) \cos 2\phi + 2r\dot{r}\dot{\phi} \sin 2\phi] + S_\theta^2(z - r^2\dot{\phi}^2 - \dot{r}^2)\}, \quad (3.1b)$$

$$h_{\times|Q} = -2\frac{Gm\eta}{c^4 R} C_\theta \quad (3.1c)$$

$$\times \{(z + r^2\dot{\phi}^2 - \dot{r}^2) \sin 2\phi - 2r\dot{r}\dot{\phi} \cos 2\phi\}. \quad (3.1d)$$

The parameter  $z$  is related to the radial coordinate of the orbit by  $z = Gm/r$ , while  $R$  is the radial distance to the source, and  $C_\theta = \cos \theta$ ,  $S_\theta = \sin \theta$  with  $\theta$  being the orbital inclination. Obviously, the temporal evolutions of  $h_{+|Q}(t)$  and  $h_{\times|Q}(t)$  require a prescription for evolving  $r, \dot{r} = dr/dt, \phi$  and  $\dot{\phi} = d\phi/dt$  in time.

In the next step, we obtain fully 3PN accurate parametric expressions for the dynamical variables appearing in the above expressions for  $h_{+|Q}(t)$  and  $h_{\times|Q}(t)$ . This requires parametric expressions not only for  $r$  and  $\phi$ , available in the previous section, but also for  $\dot{r}$  and  $\dot{\phi}$ . We obtain 3PN accurate parametric expressions for  $\dot{r}$  and  $\dot{\phi}$  by noting that  $\dot{r} = (dr/du) \times (du/dt)$  and  $\dot{\phi} = (d\phi/dv) \times (dv/du) \times (du/dt)$ . These expressions are provided in terms of a certain gauge-invariant dimensionless PN expansion parameter  $\xi = \frac{Gm\eta}{c^3 P}$ , where  $n = \frac{2\pi}{P}$  as defined in Eq. (2.36c), the time eccentricity  $e_t$  and the eccentric anomaly  $u$ . The dynamical

variables have to be derived carefully, as we introduced scaled coordinates in the previous section. Our particular choice of variables is influenced by the ease with which we can specify various initial conditions during the numerical construction of GW templates. To obtain 3PN accurate temporal evolutions of  $r, \dot{r}, \phi$  and  $\dot{\phi}$ , we also need to express the right-hand side of the 3PN accurate Kepler equation in terms of  $\xi$  and  $e_t$ .

The third step involves including the effects of GW emission during hyperbolic passages. This is accomplished by providing differential equations for  $d\xi/dt$  and  $de_t/dt$ , whose derivation is influenced by Refs. [23,24]. These equations, as expected, incorporate radiation reaction effects entering the orbital dynamics at 2.5PN and 3.5PN orders. Through a numerical solution of the Kepler equation along with these two coupled differential equations for  $\xi$  and  $e_t$ , we obtain the fully 3.5PN accurate temporal evolution for  $r, \dot{r}, \phi$  and  $\dot{\phi}$ . This enables us to construct  $h_{+|Q}(t)$  and  $h_{\times|Q}(t)$  for compact binaries in 3.5PN accurate hyperbolic orbits. Finally, we provide a 3PN accurate expression for  $\xi$  in terms of a certain PN accurate gauge-dependent impact parameter  $b$  and time eccentricity  $e_t$ , as it is very convenient to characterize hyperbolic orbits through their impact parameters and eccentricities. Thus, we obtain ready-to-use GW templates for compact binaries in PN accurate hyperbolic orbits.

In the following, we provide explicit expressions for various dynamical variables in terms of  $\xi, e_t$  and  $u$  that will be required for obtaining  $h_{+|Q}(t)$  and  $h_{\times|Q}(t)$ . For the sake of readability, we will only explicitly list the 1PN accurate expressions for these dynamical variables, given in terms of  $\xi, e_t$  and  $u$  as

$$r(u) = \frac{Gm}{c^2} \frac{1}{\xi^{2/3}} (e_t \cosh u - 1) \times \left\{ 1 + \xi^{2/3} \frac{2\eta - 18 - (6 - 7\eta)e_t \cosh u}{6(e_t \cosh u - 1)} \right\}, \quad (3.2a)$$

$$\dot{r}(u) = \xi^{1/3} \frac{ce_t \sinh u}{e_t \cosh u - 1} \left\{ 1 - \xi^{2/3} \frac{6 - 7\eta}{6} \right\}, \quad (3.2b)$$

$$\phi(u) - \phi_0 = 2 \arctan \left[ \left( \frac{e_\phi + 1}{e_\phi - 1} \right)^{1/2} \tanh u/2 \right] \times \left\{ 1 + \xi^{2/3} \frac{3}{e_t^2 - 1} \right\}, \quad (3.2c)$$

$$\dot{\phi}(u) = \frac{n\sqrt{e_t^2 - 1}}{(e_t \cosh u - 1)^2} \times \left\{ 1 - \xi^{2/3} \frac{[3 - (4 - \eta)e_t^2 + (1 - \eta)e_t \cosh u]}{(e_t^2 - 1)(e_t \cosh u - 1)} \right\}. \quad (3.2d)$$

The lengthy 3PN accurate versions of these expressions are provided in Appendix B.

It should be obvious that temporal evolutions for the 3PN version of the above equations, namely Eqs. (3.2), require a 3PN accurate Kepler equation in terms of  $\xi$  and  $e_t$  that connects  $l$  and  $u$ . This 3PN accurate equation in MH gauge is given by

$$l_{3\text{PN}} = \frac{\xi^2}{6720(e_t^2 - 1)^{3/2}(e_t \cosh u - 1)} \{35\nu\{96e_t^2[\eta(11\eta - 29) + 30] + \eta(960\eta + 123\pi^2 - 13184) + 8640\} \\ \times (e_t \cosh u - 1) + 840e_t\sqrt{e_t^2 - 1}(\eta - 4) \sinh u[e_t(\eta - 15)\eta \cos \nu + 24\eta - 60] + e_t \sin \nu(e_t \cosh u - 1) \\ \times [\eta\{70e_t^2[\eta(39\eta - 239) + 7] - 4[70\eta(\eta + 222) - 35967] - 4305\pi^2\} + 70e_t\eta\{e_t[\eta(13\eta - 73) + 23] \\ \times \cos 2\nu + 12[\eta(3\eta - 49) + 116] \cos \nu\} + 67200]\}. \quad (3.3e)$$

The above equation allows us to adapt Mikkola's method, developed to numerically solve the classical Kepler equation for hyperbolic orbits as detailed in Sec. 4 of Ref. [36]. Mikkola's very efficient and computationally inexpensive approach approximates the classical Kepler equation as a cubic polynomial in an auxiliary variable  $s(u)$ , finding its roots and substantially reducing the error of the initial guess through a fourth-order extension of Newton's method. We employ Mikkola's procedure in an iterative manner to tackle PN corrections to the classic Kepler equation appearing at 2PN and 3PN orders. It should be noted that our 3PN accurate Kepler equation is identical to the classical (Newtonian) Kepler equation at 1PN order, which is only possible due to the use of the time eccentricity  $e_t$  as a parameter to specify the orbit. To solve the above 3PN accurate Kepler equation, we tackle the 1PN accurate Kepler equation, namely  $l = n(t - t_0) = e_t \sinh u - u$ , using Mikkola's original prescription and obtain a 1PN accurate expression for  $u(l)$ .

This method requires us to express  $l$  in terms of a new variable  $s' = \sinh \frac{u}{3}$ ,

$$l = e_t(3s' + 4s'^3) - 3 \ln(s' + \sqrt{1 + s'^2}), \quad (3.4)$$

and truncating it to the third order in  $s'$ ,

$$l = 3(1 - e_t)s' + \left(4e_t + \frac{1}{2}\right)s'^3. \quad (3.5)$$

This third order polynomial can be solved in a closed form, say,  $s' = s'(l; e_t)$ . To minimize the error, replacing  $s'$  to

$$\omega(l) := s'(l) + \frac{0.071s'(l)^5}{(1 + 0.45s'(l)^2)(1 + 4s'(l)^2)e_t}.$$

Now we can get the most accurate solution,

$$u(l) = l - e_t(3\omega(l) + 4\omega(l)^3). \quad (3.6)$$

The accuracy of the solution can be further improved by the use of the Newton method as noted in Ref. [36].

$$l = n(t - t_0) = l_N + l_{1\text{PN}} + l_{2\text{PN}} + l_{3\text{PN}}, \quad (3.3a)$$

$$l_N = e_t \sinh u - u, \quad (3.3b)$$

$$l_{1\text{PN}} = 0, \quad (3.3c)$$

$$l_{2\text{PN}} = \frac{\xi^{4/3}}{8\sqrt{e_t^2 - 1}}[12\nu(5 - 2\eta) - e_t(\eta - 15)\eta \sin \nu], \quad (3.3d)$$

This allows us to express numerically the 2PN and 3PN corrections that appear on the right-hand side of Eqs. (3.3) in terms of  $(\xi, e_t, l)$ . We now introduce a certain parameter  $l'$  such that  $l' = l - l_{4,6}$ , where  $l_{4,6}$  are the 2PN and 3PN corrections present in Eqs. (3.3) which are evaluated using 1PN accurate  $u(l)$ . The 3PN accurate  $u(l)$  is obtained, as expected, by solving  $l' = e_t \sinh u - u$ , once again employing Mikkola's method. In this way, we pursue an accurate and efficient solution to our 3PN accurate Kepler equation which allows us to compute the 3PN accurate temporal evolutions for the dynamical variables present in our expressions for  $h_{+|Q}(t)$  and  $h_{\times|Q}(t)$ . We note, in passing, that to obtain these 3PN accurate expressions for  $r, \dot{r}, \phi$  and  $\dot{\phi}$ , we have used unique 3PN accurate expressions that provide  $2E$  and  $h$  in terms of  $\xi$  and  $e_t$  by inverting the relevant expressions present in our parametric solution. Further, we have also employed 3PN accurate relations that provide  $e_r$  and  $e_\phi$  in terms of  $e_t, \xi$  and  $\eta$ . We are now in a position to discuss how GW emission effects are incorporated.

GW emission influences binary dynamics at 2.5PN and 3.5PN orders, and we incorporate these effects by adapting the phasing formalism developed for eccentric binaries (detailed in Refs. [23,24]) to hyperbolic encounters. This requires us to compute time derivatives of the 1PN accurate expressions for the conserved orbital energy and angular momentum of binaries in noncircular orbits, given by Eqs. (3.35) and (3.36) in Ref. [37]. These time derivatives are obtained using PN accurate equations of motion that include both conservative and reactive terms to 1PN order, e.g., given by Eq. (3.34) of Ref. [37] and Eqs. (28) and (29) of [24]. The resulting expressions for  $dE/dt$  and  $dh/dt$  are adapted for hyperbolic orbits with the help of our 1PN accurate parametric expressions for the dynamical variables  $r, \dot{r}$  and  $\dot{\phi}$ , expressed in terms of  $n, e_t, u$ . Using our 1PN accurate expressions for  $n = 2\pi/P$  and  $e_t^2$  in terms of the conserved orbital energy and angular momentum,  $dE/dt$  and  $dh/dt$  then lead to the desired equations for  $dn/dt$  and  $de_t/dt$  in modified harmonic gauge:



$$\begin{aligned} \frac{dn}{dt} = & \frac{8c^6\eta\xi^{\frac{11}{3}}}{5G^2m^2\beta^6} [35(1-e_t^2) + 49\beta + 32\beta^2 + 6\beta^3 - 9\beta e_t^2] + \frac{2c^6\eta}{35\beta^9} \xi^{\frac{13}{3}} \{ \beta^6(180 - 588\eta) + \beta^5(1340 - 5852\eta) \\ & + 2\beta^4[9e_t^2(21\eta - 1) - 8589\eta + 1003] + 35\beta^3[e_t^2 \times (244\eta - 5) - 684\eta + 21] \\ & + 35\beta^2(e_t^2 - 1)[9e_t^2 \times (2\eta - 17) + 454\eta + 193] - 21\beta(e_t^2 - 1)^2(140\eta + 657) + 5880(e_t^2 - 1)^3 \}, \end{aligned} \quad (3.7a)$$

$$\begin{aligned} \frac{de_t}{dt} = & \frac{8c^3\eta(e_t^2 - 1)}{15Gm\beta^6 e_t} \xi^{\frac{8}{3}} [35(1 - e_t^2) + (49 - 9e_t^2)\beta + 17\beta^2 + 3\beta^3] \\ & - \frac{2c^3\eta}{315\beta^9 e_t} \xi^{\frac{10}{3}} \{ -17640(-1 + e_t^2)^4 + 63\beta \times (-1 + e_t^2)^3(657 + 140\eta) - 105\beta^2(-1 + e_t^2)^2[13 \times +454\eta + 9e_t^2(3 + 2\eta)] \\ & - \beta^4(-1 + e_t^2)[36825 - 53060\eta + 9e_t^2(-2169 + 560\eta)] + 6\beta^6[360 - 553\eta + e_t^2(-444 + 637\eta)] \\ & - 28\beta^3(-1 + e_t^2)[29(63 - 95\eta) + e_t^2(-1767 + 1105\eta)] + \beta^5[10215 - 18088\eta + e_t^2 \times (-12735 + 20608\eta)] \}, \end{aligned} \quad (3.7b)$$

where  $\beta = e_t \cosh u - 1$ . We have verified that these expressions can also be obtained by the usual calculations based on balance arguments. In this approach, one differentiates our 1PN accurate expressions for  $n$  and  $e_t$  while using 1PN accurate expressions for the far-zone fluxes, given for example by Eqs. (17) and (18) of Ref. [38], to replace the time derivatives of the conserved energy and angular momentum variables. The resulting expressions for  $dn/dt$  and  $de_t/dt$ , adapted for 1PN accurate

hyperbolic orbits, were found to be identical to Eqs. (3.7a) and (3.7b).

It is rather convenient to characterize hyperbolic binaries in terms of an impact parameter  $b$ , as these GW events are qualitatively similar to scattering processes. We define a PN accurate impact parameter  $b$  such that  $bv_\infty = |\mathbf{r} \times \mathbf{v}|$  when  $|\mathbf{r}| \rightarrow \infty$ , while  $v_\infty$  stands for the relative velocity at infinity [37]. The explicit 3PN accurate expression for  $b$  in terms of  $\xi$  and  $e_t$  in modified harmonic gauge reads

$$\begin{aligned} b = & \frac{Gm\sqrt{e_t^2 - 1}}{c^2} \xi^{\frac{2}{3}} \left\{ 1 - \xi^{\frac{2}{3}} \left( \frac{\eta - 1}{e_t^2 - 1} + \frac{7\eta - 6}{6} \right) + \xi^{\frac{4}{3}} \left[ 1 - \frac{7}{24}\eta + \frac{35}{72}\eta^2 + \frac{3 - 16\eta}{2(e_t^2 - 1)} + \frac{7 - 12\eta - \eta^2}{2(e_t^2 - 1)^2} \right] \right. \\ & + \xi^2 \left[ -\frac{2}{3} + \frac{87}{16}\eta - \frac{437}{144}\eta^2 + \frac{49}{1296}\eta^3 + \frac{36 - 378\eta + 140\eta^2 + 3\eta^3}{24(e_t^2 - 1)} \right. \\ & + \frac{1}{6720(e_t^2 - 1)^2} \{ 248640 + (-880496 + 12915\pi^2)\eta + 40880\eta^2 + 3920\eta^3 \} \\ & \left. \left. + \frac{1}{1680(e_t^2 - 1)^3} \{ 73080 + (-228944 + 4305\pi^2)\eta + 47880\eta^2 + 840\eta^3 \} \right] \right\}. \end{aligned} \quad (3.8)$$

At 1PN order, we are in agreement with Ref. [29]. This variable is essentially invoked to allow for an easy visualization of the trajectories of hyperbolic binaries.

With the above inputs, it is possible to obtain temporally evolving Newtonian (quadrupolar) GW polarization states,  $h_{+|Q}(t)$  and  $h_{\times|Q}(t)$ , associated with compact binaries in 3.5PN accurate hyperbolic orbits. It is convenient to numerically solve a system of three coupled differential equations, namely  $dn/dt$ ,  $de_t/dt$  and  $dl/dt = n$ . The resulting values of parameters  $n$ ,  $e_t$  and  $l$  at a given epoch are then employed to obtain a 3PN accurate value for  $u(l)$  by the application of Mikkola's method as described above. With a knowledge of  $n$ ,  $e_t$ ,  $l$  and  $u$ , we can then evaluate our 3PN accurate expressions for  $r$ ,  $\dot{r}$ ,  $\phi$  and  $\dot{\phi}$ . Thus, we are able to numerically provide  $h_{+|Q}(t)$  and  $h_{\times|Q}(t)$  from compact binaries in 3.5PN accurate hyperbolic orbits. In

the following, we discuss plots that demonstrate the approach and point out a feature of the waveforms previously not mentioned in the literature.

In Fig. 1, we plot scaled quadrupolar GW polarization states,  $H_{+|Q}(l)$  and  $H_{\times|Q}(l)$ , for hyperbolic passages with  $b \sim 30Gm/c^2$  for compact binaries having  $m = 20 M_\odot$  and  $\eta = \frac{1}{4}$  ( $q = 1$ ), while allowing  $e_t$  to take three different values. Here,  $H_{+|Q}(l)$  and  $H_{\times|Q}(l)$  denote waveforms that have been scaled by  $Gm/c^2 R$ . We observe, as expected, the linear memory effect for the cross polarization [29]. We display in Fig. 2 the trajectories of compact binaries under the influence of Newtonian and fully 3.5PN accurate orbital dynamics (respectively in black and red) and their associated  $H_{\times|Q}(l)$ . For these, we have chosen  $e_t = 1.1$  while we let the impact parameter  $b$  take two different values, namely,  $\sim 50Gm/c^2$  and  $\sim 106Gm/c^2$ . These particular



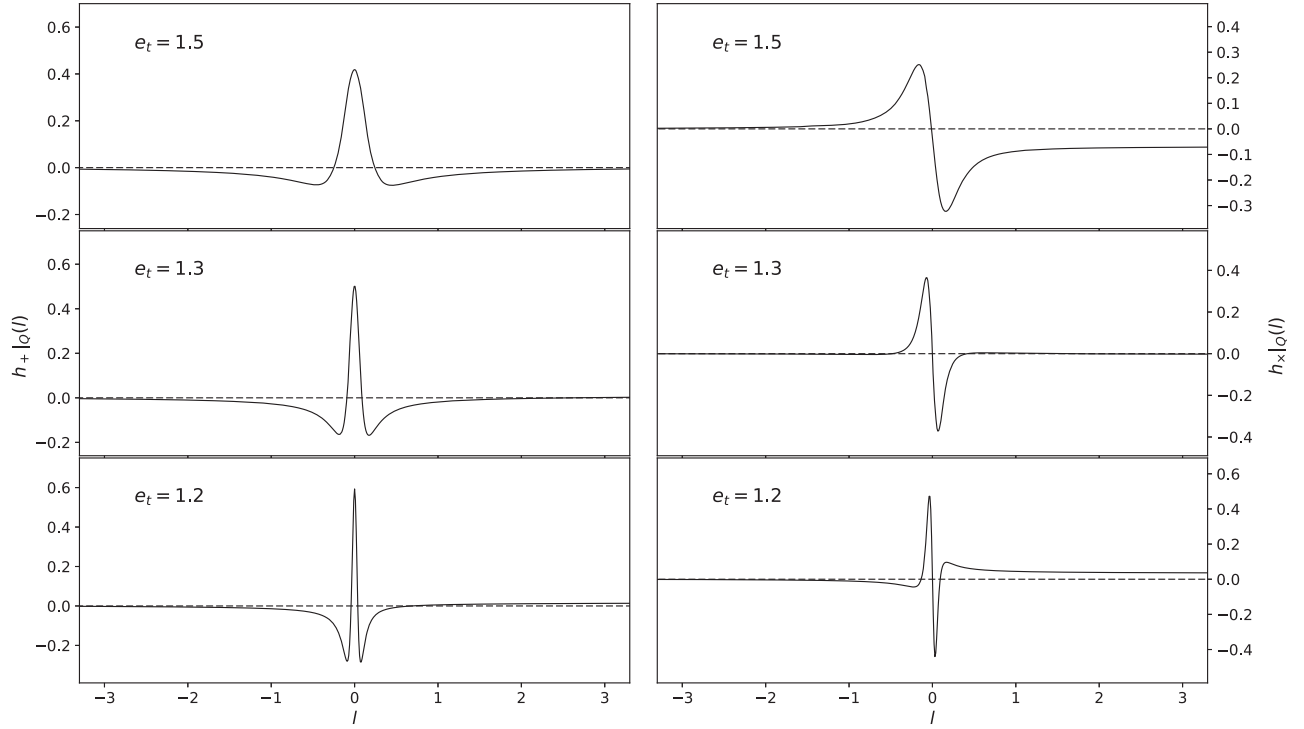


FIG. 1. Scaled  $H_{+|Q}(l)$  and  $H_{\times|Q}(l)$  plots for nonspinning compact binaries with total mass  $m = 20 M_{\odot}$  and mass ratio  $q = 1$ . We let the eccentricity  $e_t$  take three values 1.5, 1.3 and 1.2, while choosing an impact parameter  $b \sim 30 Gm/c^2$  and inclination angle  $\theta = \frac{\pi}{4}$ . We observe the expected linear memory effect in the cross polarization state.

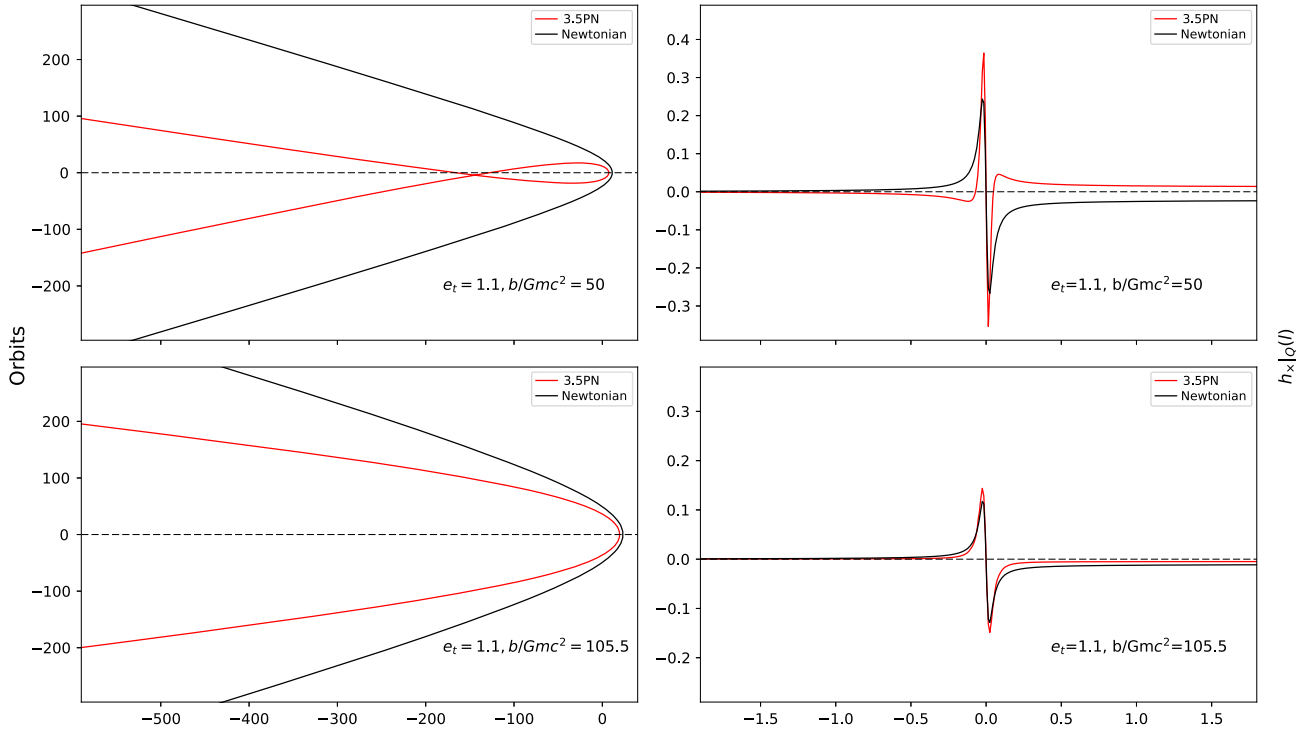


FIG. 2. Trajectories and the associated scaled  $H_{\times|Q}(l)$  for hyperbolic compact binaries, with a choice of two different impact parameters  $b$ , eccentricity  $e_t = 1.1$ , total mass  $m = 20 M_{\odot}$ , mass ratio  $q = 1$ , and inclination angle  $\theta = \frac{\pi}{4}$ . For the trajectories, we adopt the geometric unit system. Newtonian and 3.5PN accurate hyperbolic orbits are denoted by black and red lines, respectively. The orbital trajectory of the relativistic system is clearly different, especially for hyperbolic passages with smaller  $b$  values, which is attributed to the advance of periastron. Relativistic effects also change the nature of the waveforms, as evident from the associated  $h_{\times|Q}(l)$  plots.

$v$  values were chosen to highlight the effect of PN corrections compared to the familiar Newtonian hyperbolic orbit. We observe that the periastron advance forces the 3.5PN accurate orbital trajectory to cross its earlier path, a feature which is absent in the Newtonian system. Additionally, this feature disappears for large impact parameter values. This is expected, as the periastron advance is small for configurations with a large impact parameter, which results in Newtonian-like trajectories. We have also verified that the numerical values of 3PN accurate  $\Phi/2\pi$  indeed remain close to their counterparts originating from 1PN accurate expression for  $\Phi/2\pi$ ; this ensures that the crossing of the trajectory is a physical effect.

#### IV. CONCLUSIONS

In this paper, we provided “ready-to-use” time-domain GW polarization templates for compact binaries moving in fully 3.5PN accurate hyperbolic orbits. A crucial input for constructing these waveforms is our *ab initio* derivation of 3PN accurate Keplerian-type parametric solutions for compact binaries in hyperbolic orbits. Our effort extended the classic 1PN result of Damour and Deruelle, obtained by the argument of analytic continuation to 3PN order [17]. Additionally, we provided two critical checks to verify the correctness of our solution and its lengthy 3PN accurate expressions. We incorporated the effects of GW emission, occurring at 2.5PN and 3.5PN orders in the orbital dynamics, by adapting for hyperbolic orbits GW phasing formalism for eccentric inspirals, detailed in Refs. [23,24]. This is how we constructed our PN accurate GW templates, namely temporally evolving GW polarization states, for hyperbolic encounters.

The present effort should be useful in a number of ongoing investigations. Our templates are being implemented in the LSC Algorithm Library Suite (LALSuite) [1]. This is to explore the possibility of searching for the presence of such GW events in the interferometric data streams in the near future. The following plausible astrophysical considerations should provide motivations for initiating such efforts. It was pointed out that such

encounters involving neutron stars can give rise to certain resonant shattering flares in the electromagnetic sector due to strong tidal interactions between neutron stars during hyperbolic encounters, though event rates are expected to be low [35]. Very recently, it was argued that aLIGO relevant GW burst events may occur during hyperbolic encounters of primordial black holes in dense clusters [31]. Therefore, it should be of some interest to explore the search sensitivity and the possible false alarm rates of hyperbolic GW events by adapting such an effort for eccentric inspirals [39].

The present computation will be crucial to obtain fully 3PN accurate expressions for radiated energy and angular momentum fluxes associated with hyperbolic encounters which are only available to 1PN order [37,38]. Currently, these computations are being extended to 3PN order [40]. This investigation is expected to complement efforts that focus on the scatterings of test particles by black hole space-time [41–43]. It will also be desirable to adapt Refs. [44–46] for exploring our GW burst signals using the framework of effective-one-body formalism.

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#### APPENDIX A: GENERALIZED QUASI-KEPLERIAN PARAMETRIZATION FOR HYPERBOLIC COMPACT BINARIES IN ADM-TYPE GAUGE

We follow exactly the same procedure, detailed in Sec. II B, while using Eqs. (A1) and (A2) of Ref. [20] to derive the 3PN accurate hyperbolic parametrization in an ADM-type gauge. The third post-Newtonian accurate generalized quasi-Keplerian parametrization, in ADM coordinates, for hyperbolic compact binaries is given by

$$r = a_r(e_r \cosh u - 1), \quad (\text{A1a})$$

$$\frac{2\pi}{P}(t - t_0) = e_t \sinh u - u + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\nu + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)\sin \nu + \frac{h_{6t}}{c^6}\sin 2\nu + \frac{i_{6t}}{c^6}\sin 3\nu, \quad (\text{A1b})$$

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = \nu + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right)\sin 2\nu + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right)\sin 3\nu + \frac{h_{6\phi}}{c^6}\sin 4\nu + \frac{i_{6\phi}}{c^6}\sin 5\nu, \quad (\text{A1c})$$

where  $\nu = 2 \tanh^{-1}[(\frac{e_\phi + 1}{e_\phi - 1})^{1/2} \tan \frac{u}{2}]$ . The explicit 3PN accurate expressions for the orbital elements and functions of the generalized quasi-Keplerian parametrization, in ADM coordinates, read

$$\begin{aligned}
a_r = & \frac{1}{(2E)} \left\{ 1 + \frac{(2E)}{4c^2} (7 - \eta) + \frac{(2E)^2}{16c^4} \left[ (1 + 10\eta + \eta^2) + \frac{1}{(2Eh^2)} (68 - 44\eta) \right] \right. \\
& + \frac{(2E)^3}{192c^6} \left[ -3 + 9\eta + 6\eta^2 - 3\eta^3 + \frac{1}{(2Eh^2)} (864 + (-3\pi^2 - 2212)\eta + 432\eta^2) \right. \\
& \left. \left. + \frac{1}{(2Eh^2)^2} (6432 - (13488 - 240\pi^2)\eta + 768\eta^2) \right] \right\}, \tag{A2a}
\end{aligned}$$

$$\begin{aligned}
e_r^2 = & 1 + 2Eh^2 + \frac{(2E)}{4c^2} \{-24 + 4\eta + 5(-3 + \eta)(2Eh^2)\} \\
& + \frac{(2E)^2}{8c^4} \left\{ 52 + 2\eta + 2\eta^2 + (80 - 55\eta + 4\eta^2)(2Eh^2) + \frac{8}{(2Eh^2)} (-17 + 11\eta) \right\} \\
& + \frac{(2E)^3}{192c^6} \left\{ 768 + 6\eta\pi^2 + 344\eta + 216\eta^2 + 3(2Eh^2)(-1488 + 1556\eta - 319\eta^2 + 4\eta^3) \right. \\
& \left. - \frac{4}{(2Eh^2)} (588 - 8212\eta + 177\eta\pi^2 + 480\eta^2) - \frac{192}{(2Eh^2)^2} (134 - 281\eta + 5\eta\pi^2 + 16\eta^2) \right\}, \tag{A2b}
\end{aligned}$$

$$n = (2E)^{3/2} \left\{ 1 + \frac{(2E)}{8c^2} (-15 + \eta) + \frac{(2E)^2}{128c^4} [555 + 30\eta + 11\eta^2] + \frac{(2E)^3}{1024c^6} [653 + 111\eta + 7\eta^2 + 3\eta^3] \right\}, \tag{A2c}$$

$$\begin{aligned}
e_t^2 = & 1 + 2Eh^2 + \frac{(2E)}{4c^2} \{8 - 8\eta + (17 - 7\eta)(2Eh^2)\} \\
& + \frac{(2E)^2}{8c^4} \left\{ 8 + 4\eta + 20\eta^2 + (2Eh^2)(112 - 47\eta + 16\eta^2) + \frac{4}{(2Eh^2)} (-17 + 11\eta) \right\} \\
& + \frac{(2E)^3}{192c^6} \left\{ 24(2 - 5\eta)(-23 + 10\eta + 4\eta^2) - 15(-528 + 200\eta - 77\eta^2 + 24\eta^3)(2Eh^2) \right. \\
& \left. - \frac{2}{(2Eh^2)} (6732 + 117\eta\pi^2 - 12508\eta + 2004\eta^2) - \frac{96}{(2Eh^2)^2} (134 - 281\eta + 5\eta\pi^2 + 16\eta^2) \right\}, \tag{A2d}
\end{aligned}$$

$$f_{4t} = \frac{3(2E)^2}{2} \left\{ \frac{5 - 2\eta}{\sqrt{(2Eh^2)}} \right\}, \tag{A2e}$$

$$f_{6t} = \frac{(2E)^3}{192(2Eh^2)^{\frac{3}{2}}} \{ (10080 + 123\eta\pi^2 - 13952\eta + 1440\eta^2) + (2Eh^2)36(95 - 55\eta + 18\eta^2) \}, \tag{A2f}$$

$$g_{4t} = -\frac{1}{8} \frac{(2E)^2}{\sqrt{(2Eh^2)}} \{ (4 + \eta)\eta\sqrt{(1 + 2Eh^2)} \}, \tag{A2g}$$

$$\begin{aligned}
g_{6t} = & \frac{(2E)^3}{192(2Eh^2)^{\frac{3}{2}}\sqrt{1 + 2Eh^2}} \{ 3(2Eh^2)^2\eta(23\eta^2 - 4\eta - 64) + (2Eh^2)(105\eta^3 + 627\eta^2 + (3\pi^2 - 4232)\eta + 1728) \\
& + 33\eta^3 + 600\eta^2 + (3\pi^2 - 4148)\eta + 1728 \}, \tag{A2h}
\end{aligned}$$

$$h_{6t} = \frac{(2E)^3}{32} \eta \left\{ \frac{(1 + 2Eh^2)}{(2Eh^2)^{3/2}} (23 + 12\eta + 6\eta^2) \right\}, \tag{A2i}$$

$$i_{6t} = \frac{13(2E)^3}{192} \eta^3 \left( \frac{1 + 2Eh^2}{2Eh^2} \right)^{3/2}, \tag{A2j}$$

$$\begin{aligned} \Phi = 2\pi & \left\{ 1 + \frac{3}{c^2 h^2} + -\frac{3(2E)^2}{4(2Eh^2)^2 c^4} [-35 + 10\eta + (2Eh^2)(-5 + 2\eta)] \right. \\ & + \frac{(2E)^3}{128c^6(2Eh^2)^3} [36960 + (615\pi^2 - 40000)\eta + 1680\eta^2 + (2Eh^2)(10080 + 123\eta\pi^2 - 13952\eta + 1440\eta^2) \\ & \left. + (2Eh^2)^2(120 - 120\eta + 96\eta^2)] \right\}, \end{aligned} \quad (\text{A2k})$$

$$f_{4\phi} = \frac{(2E)^2(1 + 2Eh^2)}{8(2Eh^2)^2} \eta(1 - 3\eta), \quad (\text{A2l})$$

$$\begin{aligned} f_{6\phi} = \frac{(2E)^3}{256(2Eh^2)^3} & \{256 + (-1076 + 49\pi^2)\eta - 384\eta^2 - 40\eta^3 + 4(2Eh^2)^2\eta(-11 - 40\eta + 24\eta^2) \\ & + (2Eh^2)(256 + (-1192 + 49\pi^2)\eta - 336\eta^2 + 80\eta^3)\}, \end{aligned} \quad (\text{A2m})$$

$$g_{4\phi} = -\frac{3(2E)^2}{32} \frac{\eta^2}{(2Eh^2)^2} (1 + 2Eh^2)^{3/2}, \quad (\text{A2n})$$

$$g_{6\phi} = \frac{(2E)^3}{768} \frac{\sqrt{(1 + 2Eh^2)}}{(2Eh^2)^3} \eta \{220 + 3\pi^2 + 96\eta + 45\eta^2 + 3(2Eh^2)^2\eta(-9 + 26\eta) + (2Eh^2)(220 + 3\pi^2 + 312\eta + 150\eta^2)\}, \quad (\text{A2o})$$

$$h_{6\phi} = \frac{(2E)^3(1 + 2Eh^2)^2}{128(2Eh^2)^3} \eta(5 + 28\eta + 10\eta^2), \quad (\text{A2p})$$

$$i_{6\phi} = \frac{5(2E)^3}{256} \frac{\eta^3}{(2Eh^2)^3} (1 + 2Eh^2)^{5/2}, \quad (\text{A2q})$$

$$\begin{aligned} e_{\phi}^2 = 1 + 2Eh^2 + \frac{(2E)}{4c^2} & \{-24 + (-15 + \eta)(2Eh^2)\} + \frac{(2E)^2}{16c^4(2Eh^2)} \{-408 + 232\eta + 15\eta^2 + (2Eh^2)(-32 + 176\eta + 18\eta^2) \\ & + (2Eh^2)^2(160 - 30\eta + 2\eta^2)\} - \frac{(2E)^3}{384c^6(2Eh^2)^2} \{3(27776 + (-65436 + 1325\pi^2)\eta + 3440\eta^2 - 70\eta^3) \\ & + 36(2Eh^2)^3(248 - 80\eta + 13\eta^2 + \eta^3) + 6(2Eh^2)(2456 + (-26860 + 581\pi^2)\eta + 2689\eta^2 + 10\eta^3) \\ & + (2Eh^2)^2(-16032 + (2764 + 3\pi^2)\eta + 4536\eta^2 + 234\eta^3)\}. \end{aligned} \quad (\text{A2r})$$

## APPENDIX B: FULLY 3PN ACCURATE EXPRESSIONS FOR THE DYNAMICAL VARIABLES THAT APPEAR IN THE EXPRESSIONS FOR $h_+|_Q(l)$ AND $h_{\times}|_Q(l)$

Extending the results we listed in Eqs. (3.2), we provide 3PN accurate expressions for  $r$ ,  $\dot{r}$ ,  $\phi$  and  $\dot{\phi}$  in terms of  $\xi$ ,  $e_t$  and  $\eta$  in modified harmonic gauge. The orbital separation reads

$$r = r_N + r_{1PN} + r_{2PN} + r_{3PN}, \quad (\text{B1a})$$

where

$$r_N = \frac{Gm}{c^2} \frac{1}{\xi^{2/3}} (e_t \cosh u - 1), \quad (\text{B1b})$$

$$r_{1PN} = r_N \times \frac{\xi^{2/3}}{6(e_t \cosh u - 1)} [(7\eta - 6)e_t \cosh u + 2(\eta - 9)], \quad (\text{B1c})$$

$$r_{2\text{PN}} = r_{\text{N}} \times \frac{\xi^{4/3}}{72(e_t^2 - 1)(e_t \cosh u - 1)} [(e_t^2 - 1)e_t(35\eta^2 - 231\eta + 72) \cosh u - 2e_t^2(4\eta^2 + 15\eta + 36) + 8\eta^2 + 534\eta - 216], \quad (\text{B1d})$$

$$r_{3\text{PN}} = r_{\text{N}} \times \frac{\xi^2}{181440(e_t^2 - 1)^2(e_t \cosh u - 1)} \{280e_t^4(16\eta^3 + 90\eta^2 - 81\eta + 432) + 140(e_t^2 - 1)^2e_t(49\eta^3 - 3933\eta^2 + 7047\eta - 864) \cosh u - e_t^2[8960\eta^3 + 3437280\eta^2 + 81(1435\pi^2 - 134336)\eta + 3144960] + 4480\eta^3 - 761040\eta^2 - 348705\pi^2\eta + 12143736\eta - 4233600\}. \quad (\text{B1e})$$

The angular variable of the 3PN accurate motion is given by

$$\phi = \phi_{\text{N}} + \phi_{1\text{PN}} + \phi_{2\text{PN}} + \phi_{3\text{PN}}, \quad (\text{B2a})$$

where

$$\phi_{\text{N}} = \nu, \quad (\text{B2b})$$

$$\phi_{1\text{PN}} = \frac{\xi^{2/3}}{(e_t^2 - 1)(e_t \cosh u - 1)} \left[ e_t \sqrt{e_t^2 - 1} (4 - \eta) \sinh u + 3\eta(e_t \cosh u - 1) \right], \quad (\text{B2c})$$

$$\begin{aligned} \phi_{2\text{PN}} = & \frac{\xi^{4/3}}{192(e_t^2 - 1)^{5/2}(e_t \cosh u - 1)^2} \{e_t(e_t^2 - 1)[2\{e_t^2[384 - \eta(7\eta + 275)] + 4[\eta(\eta + 137) - 792]\} \sinh u \\ & + e_t\{e_t^2[\eta(55\eta - 109) + 384] - 4[\eta(13\eta + 41) - 600]\} \sinh 2u + 6\sqrt{e_t^2 - 1}(e_t \cosh u - 1)^2 \\ & \times \{e_t^3(1 - 3\eta)\eta \sin 3\nu - 8\nu[e_t^2(26\eta - 51) + 28\eta - 78] + 4e_t^2[(19 - 3\eta)\eta + 1] \sin 2\nu\}\}, \end{aligned} \quad (\text{B2d})$$

$$\begin{aligned} \phi_{3\text{PN}} = & \frac{\xi^2}{53760(e_t^2 - 1)^{7/2}(e_t \cosh u - 1)^3} \left( \sqrt{e_t^2 - 1}(e_t \cosh u - 1)^3(2e_t^2\{[280e_t^2\{\eta[\eta(93\eta - 781) + 886] + 24\} \right. \\ & + \eta\{32[35\eta(9\eta - 395) + 36877] - 30135\pi^2\} + 84000\} \sin 2\nu + e_t\eta[\{35e_t^2[\eta(129\eta - 137) + 33] \\ & + 4[35 \times \eta(51\eta - 727) + 28302] - 4305\pi^2\} \sin 3\nu + 35e_t\{3e_t[5(\eta - 1)\eta + 1] \sin 5\nu + 4[3\eta(5\eta - 19) + 82] \sin 4\nu\}]\} \\ & + 420\nu\{16(65e_t^4 + 320e_t^2 + 56)\eta^2 + [123\pi^2(e_t^2 + 4) - 32(55e_t^4 + 870e_t^2 + 793)]\eta + 96(26e_t^4 + 293e_t^2 + 190)\} \\ & + e_t(e_t^2 - 1) \sinh u \{-70e_t^6\eta[\eta(71\eta + 61) - 639] + 1680e_t^4(\eta - 4) \cosh^2 u \{3e_t\eta(3\eta - 1) \cos 3\nu \\ & + 8[\eta(3\eta - 19) - 1] \cos 2\nu\} + e_t^4[\eta\{4[70\eta(125\eta - 507) - 462853] - 4305\pi^2\} + 3933440] + 1680e_t^2(\eta - 4) \\ & \times \{3e_t\eta(3\eta - 1) \cos 3\nu + 8[\eta(3\eta - 19) - 1] \cos 2\nu\} + e_t^2\{6\eta[140\eta(25\eta - 397) + 1435\pi^2 - 917424] \\ & + 7947520\} + 4e_t \cosh u \{-70e_t^4\{\eta[\eta(39\eta - 719) + 2279] - 3072\} + 840e_t^2(\eta - 4)\{3e_t(1 - 3\eta)\eta \cos 3\nu \\ & + 8[(19 - 3\eta)\eta + 1] \cos 2\nu\} + e_t^2[\eta\{8[35(232 - 53\eta)\eta + 186959] + 4305\pi^2\} - 2983680] - 20[323904 \\ & - \eta\{4[7\eta(\eta + 45) + 56013] - 861\pi^2\}]\} + e_t^2\{-70e_t^4\eta[\eta(71\eta + 61) - 639] + e_t^2[\eta\{4[35\eta(229\eta - 1173) \\ & - 384978] - 4305\pi^2\} + 3646720] + 20[\eta\{861\pi^2 - 4[14\eta(9\eta - 25) + 54025]\} + 280000]\} \cosh 2u \\ & \left. + 40[\eta(1456\eta + 861\pi^2 - 253508) + 396480]\right\}, \end{aligned} \quad (\text{B2e})$$

with  $\nu = 2 \arctan(\sqrt{\frac{e_t^2 + 1}{e_t^2 - 1}} \tanh \frac{u}{2})$  as before. Furthermore, we require explicit expressions for the first time derivatives  $\dot{r}$  and  $\dot{\phi}$  to compute GW waveforms from binaries in hyperbolic orbits, namely,

$$\dot{r} = \dot{r}_{\text{N}} + \dot{r}_{1\text{PN}} + \dot{r}_{2\text{PN}} + \dot{r}_{3\text{PN}}, \quad (\text{B3a})$$



$$\dot{r}_N = \xi^{1/3} \frac{ce_t \sinh u}{e_t \cosh u - 1}, \quad (\text{B3b})$$

$$\dot{r}_{1\text{PN}} = \dot{r}_N \times \frac{\xi^{2/3}}{6} (7\eta - 6), \quad (\text{B3c})$$

$$\dot{r}_{2\text{PN}} = \dot{r}_N \times \frac{\xi^{4/3}}{72(e_t \cosh u - 1)^2} \{9e_t(\eta - 15)\eta \cos \nu + e_t[7\eta(5\eta - 33) + 72] \cosh u (e_t \cosh u - 2) + 5\eta(7\eta - 3) - 468\}, \quad (\text{B3d})$$

$$\begin{aligned} \dot{r}_{3\text{PN}} = \dot{r}_N \times \frac{\xi^2}{181440(e_t^2 - 1)^{3/2}(e_t \cosh u - 1)^3} \bigg\{ & 3780(e_t^2 - 1)^{3/2}(7\eta - 6)(e_t \cosh u - 1)[e_t(\eta - 15)\eta \cos \nu + 24\eta - 60] \\ & + 140(e_t^2 - 1)^{3/2}(49\eta^3 - 3933\eta^2 + 7047\eta - 864)(e_t \cosh u - 1)^3 - 27[-840(e_t^2 - 1)e_t^2\eta(\eta^2 - 19\eta + 60) \sin \nu \sinh u \\ & - 840\sqrt{e_t^2 - 1}e_t^3\eta(\eta^2 - 19\eta + 60) \cos \nu \sinh^2 u + 840\sqrt{e_t^2 - 1}e_t^3\eta(\eta^2 - 19\eta + 60) \cos \nu \times \cosh^2 u \\ & + \sqrt{e_t^2 - 1}e_t \cosh u \{e_t[35(65e_t^2 - 32)\eta^3 - 525(27e_t^2 + 88)\eta^2 + (-315e_t^2 - 4305\pi^2 + 93468)\eta + 67200] \cos \nu \\ & + 35[3e_t^3\eta(13\eta^2 - 73\eta + 23) \cos 3\nu + 24e_t^2\eta(3\eta^2 - 49\eta + 116) \cos 2\nu + 1056e_t^2\eta^2 - 2784e_t^2\eta \\ & + 2880e_t^2 + 384\eta^2 + 123\pi^2\eta - 9440\eta + 2880]\} - \sqrt{e_t^2 - 1}\{e_t[35(65e_t^2 - 8)\eta^3 - 105(135e_t^2 + 592)\eta^2 - 3 \\ & \times (105e_t^2 + 1435\pi^2 - 47956)\eta + 67200] \cos \nu + 35[3e_t^3\eta(13\eta^2 - 73\eta + 23) \cos 3\nu + 24e_t^2\eta(3\eta^2 - 49\eta + 116) \\ & \times \cos 2\nu + 480e_t^2\eta^2 + 960e_t^2\eta - 2880e_t^2 + 960\eta^2 + 123\pi^2\eta - 13184\eta + 8640]\} \bigg\}, \quad (\text{B3e}) \end{aligned}$$

as well as

$$\dot{\phi} = \dot{\phi}_N + \dot{\phi}_{1\text{PN}} + \dot{\phi}_{2\text{PN}} + \dot{\phi}_{3\text{PN}}, \quad (\text{B4a})$$

$$\dot{\phi}_N = \frac{n\sqrt{e_t^2 - 1}}{(e_t \cosh u - 1)^2}, \quad (\text{B4b})$$

$$\dot{\phi}_{1\text{PN}} = \dot{\phi}_N \times \frac{\xi^{2/3}}{(e_t^2 - 1)(e_t \cosh u - 1)} [e_t^2(\eta - 4) + e_t(\eta - 1) \cosh u - 3], \quad (\text{B4c})$$

$$\begin{aligned} \dot{\phi}_{2\text{PN}} = \dot{\phi}_N \times \frac{\xi^{4/3}}{192(e_t^2 - 1)^2(e_t \cosh u - 1)^2} \bigg\{ & -6e_t^4 \cosh^2 u [3e_t\eta(3\eta - 1) \cos 3\nu + 8(3\eta^2 - 19\eta - 1) \cos 2\nu] \\ & - e_t^2 [\times e_t^2(103\eta^2 + 131\eta - 72) - 4(25\eta^2 - 223\eta + 60)] \cosh 2u + 2e_t \cosh u [55e_t^4\eta^2 - 109e_t^4\eta + 384e_t^4 \\ & + 18e_t^3\eta \times (3\eta - 1) \cos 3\nu + 48e_t^2(3\eta^2 - 19\eta - 1) \cos 2\nu - 45e_t^2\eta^2 + 1359e_t^2\eta - 432e_t^2 - 4\eta^2 + 796\eta - 576] \\ & + 3[-7e_t^4\eta^2 - 291e_t^4\eta + 312e_t^4 - 18e_t^3\eta^2 \cos 3\nu + 6e_t^3\eta \cos 3\nu + 16e_t^2(-3\eta^2 + 19\eta + 1) \cos 2\nu \\ & + 8(e_t^2 - 1)^2e_t(\eta - 15)\eta \cos \nu + 4e_t^2\eta^2 - 476e_t^2\eta - 768e_t^2 - 256\eta + 768] \bigg\}, \quad (\text{B4d}) \end{aligned}$$

$$\begin{aligned}
\dot{\phi}_{\text{3PN}} = \dot{\phi}_{\text{N}} \times & \frac{\xi^2}{107520(e_t^2 - 1)^3(e_t \cosh u - 1)^3} (140\eta\{16[3\eta(5\eta - 19) + 82] \cos 4\nu + 15e_t[5(\eta - 1)\eta + 1] \cos 5\nu\} \\
& \times \cosh^3 u e_t^7 + 4\{3e_t\eta[-525[5(\eta - 1)\eta + 1] \cos 5\nu e_t^2 - 560[3\eta(5\eta - 19) + 82] \cos 4\nu e_t \\
& + 2\{-35[\eta \times (93\eta + 19) - 15]e_t^2 - 8[35\eta(21\eta - 344) + 13941] + 4305\pi^2\} \cos 3\nu - 4[\eta\{280(75\eta^2 - 595\eta + 436)e_t^2 \\
& + 16[35\eta(9\eta - 697) + 65879] - 30135\pi^2\} + 77280] \cos 2\nu\} \cosh^2 u e_t^4 + [70\{\eta[291\eta + 1865] - 2639] \\
& + 1344\}e_t^4 + \{\eta[140(10271 - 475\eta)\eta + 30135\pi^2 - 4218008] + 934080\}e_t^2 - 240(5521\eta + 56) \\
& + 140\eta[16 \times \eta(13\eta + 24) + 615\pi^2] \cosh 3ue_t^3 + 2\{-3\eta\{35[\eta(129\eta - 137) + 33]e_t^2 + 1540\eta(3\eta - 59) - 4305\pi^2 \\
& + 109848\} \cos 3\nu e_t^3 - 2[280\{\eta[93\eta - 781] + 886\} + 24\}e_t^2 + (-338240\eta - 30135\pi^2 + 928064)\eta + 70560] \\
& \times \cos 2\nu e_t^2 - 70(71e_t^6 - 572e_t^4 - 260e_t^2 + 32)\eta^3 - 70(61e_t^6 + 14900e_t^4 + 44228e_t^2 + 10336)\eta^2 \\
& + 6720(138e_t^4 - 377e_t^2 - 214) + [44730e_t^6 + 518508e_t^4 + 8417056e_t^2 - 4305(e_t^2 + 8)^2\pi^2 + 8203040]\eta\} \cosh 2ue_t^2 \\
& + \cosh u \times (-70\{\eta[319\eta - 8163] + 16997\} - 30144\}e_t^6 + [\eta\{55965\pi^2 - 4[35\eta(1331\eta - 56471) + 953852]\} \\
& - 7627200]e_t^4 + 2\{2[280\{\eta[93\eta - 781] + 886\} + 24\}e_t^2 + (-338240\eta - 30135\pi^2 + 928064)\eta + 70560] \\
& \times \cos 2\nu + 3e_t\eta\{35[\eta(129\eta - 137) + 33]e_t^2 + 1540\eta(3\eta - 59) - 4305\pi^2 + 109848\} \cos 3\nu\} \cosh 2ue_t^4 \\
& + 56\{\eta[10(24299 - 83\eta)\eta + 10455\pi^2 - 739322] + 130320\}e_t^2 + 2(e_t(2\{280\{\eta[57\eta - 193] - 506\} + 24\} \\
& \times e_t^4 + [\eta\{16[35\eta(243\eta - 2791) + 136754] - 30135\pi^2\} + 57120]e_t^2 + 6[\eta\{16[35\eta(9\eta - 679) + 64444] \\
& - 30135\pi^2\} + 79520]\} \cos 2\nu + 3e_t\eta\{[35[\eta(25\eta + 447) - 151]e_t^4 + \{2[35\eta(413\eta - 1669) + 58109] \\
& - 4305\pi^2\}e_t^2 + 8[70\eta(61\eta - 1015) + 83261] - 25830\pi^2\} \cos 3\nu + 70e_t\{16[3\eta(5\eta - 19) + 82] \cos 4\nu + 15e_t \\
& \times [5(\eta - 1)\eta + 1] \cos 5\nu\}) - 8(e_t^2 - 1)^2[\eta\{35[5\eta(13\eta - 81) - 9]e_t^2 - 4[70\eta(7\eta + 117) - 20217] - 4305\pi^2\} \\
& + 67200] \cos \nu)e_t + 320[\eta(7028\eta + 3444\pi^2 - 123467) + 42000])e_t + 2(-70(71e_t^4 - 542e_t^2 \\
& - 744)\eta^3 e_t^4 + (8[\eta\{35[\eta(17\eta + 507) - 2889]e_t^2 - 4[70\eta(\eta + 231) - 45417] - 4305\pi^2\} + 67200] \cos \nu \\
& \times (e_t^2 - 1)^2 + e_t(-2\{280\{\eta[57\eta - 193] - 506\} + 24\}e_t^4 + [\eta\{16[35\eta(93\eta - 1601) + 106234] - 30135\pi^2\} \\
& + 57120]e_t^2 + 64(30787\eta + 2625) + 70\eta[16\eta(9\eta - 643) - 861\pi^2]\} \cos 2\nu + e_t\eta\{3[-35[\eta(25\eta + 447) - 151] \\
& \times e_t^4 + \{4305\pi^2 - 2[35\eta(227\eta - 1707) + 59159]\}e_t^2 + 2[280(327 - 19\eta)\eta + 4305\pi^2 - 109988]\} \cos 3\nu + 70e_t \\
& \times \{-16[3\eta(5\eta - 19) + 82] \cos 4\nu - 15e_t[5(\eta - 1)\eta + 1] \cos 5\nu\}) - 840\sqrt{e_t^2 - 1}(\eta - 4)\{64e_t[\eta(3\eta - 19) \\
& - 1] \cos \nu(e_t \cosh u - 1)^2 + 36e_t^2\eta(3\eta - 1) \cos 2\nu(e_t \cosh u - 1)^2 + \eta[(19\eta + 111)e_t^4 + 9(3\eta - 1)(e_t \cosh 2u \\
& - 4 \cosh u)e_t^3 + (70\eta - 258)e_t^2 - 8(\eta - 15)]\} \sin \nu \sinh u)e_t - 70(61e_t^8 + 11454e_t^6 + 57640e_t^4 + 45184e_t^2 \\
& + 1536)\eta^2 + 6720(34e_t^6 - e_t^4 - 188e_t^2 - 600) + [44730e_t^8 + 263008e_t^6 + 4781712e_t^4 + 17066880e_t^2 - 4305(e_t^6 \\
& + 10e_t^4 + 84e_t^2 + 40)\pi^2 + 6482560]\eta). \tag{B4e}
\end{aligned}$$

### APPENDIX C: RELATIONS BETWEEN COEFFICIENTS IN THE PARAMETRIZATION OF $t - t_0$ AND $\phi - \phi_0$

Our 3PN accurate Keplerian-type parametric solution, derived from first principles, relies on explicit expressions for certain coefficients  $c_i$ ,  $c'_i$  and  $e_i$ ,  $e'_i$  to parametrize the radial and angular motion, respectively. In Sec. II B, we used the following explicit relations between coefficients  $c_i$  and  $c'_i = b_i/(a_r^{i-1}\sqrt{-s_+s_-})$  to obtain the parametric solution for  $t - t_0$  in Eq. (2.28) from Eq. (2.27):

$$c_0 = c'_0 e_r, \tag{C1a}$$

$$c_1 = c'_0 - c'_1, \tag{C1b}$$

$$c_2 = \frac{c'_2}{(e_r^2 - 1)^{1/2}} + \frac{c'_3}{(e_r^2 - 1)^{3/2}} + \frac{c'_4(e_r^2 + 2)}{2(e_r^2 - 1)^{5/2}} + \frac{c'_5}{(e_r^2 - 1)^{7/2}} \left(1 + \frac{3e_r^2}{2}\right), \quad (\text{C1c})$$

$$c_3 = \frac{c'_3 e_r}{(e_r^2 - 1)^{3/2}} + \frac{2c'_4 e_r}{(e_r^2 - 1)^{5/2}} + \frac{c'_5}{(e_r^2 - 1)^{7/2}} \left(3e_r + \frac{3}{4}e_r^3\right), \quad (\text{C1d})$$

$$c_4 = \frac{c'_4 e_r^2}{4(e_r^2 - 1)^{5/2}} + \frac{3c'_5 e_r^2}{4(e_r^2 - 1)^{7/2}}, \quad (\text{C1e})$$

$$c_5 = \frac{c'_5 e_r^3}{12(e_r^2 - 1)^{7/2}}. \quad (\text{C1f})$$

Also in Sec. II B, the parametric solution for  $\phi - \phi_0$  in Eq. (2.31) was obtained from Eq. (2.30) by using explicit relations between the coefficients  $e_i$  and  $e'_i = d_i/(a_r^{i+1} \sqrt{-s_+ s_-})$ , namely,

$$e_0 = \frac{e'_0}{(e_r^2 - 1)^{1/2}} + \frac{e'_1}{(e_r^2 - 1)^{3/2}} + \frac{e'_2(e_r^2 + 2)}{2(e_r^2 - 1)^{5/2}} + \frac{e'_3(3e_r^2 + 2)}{2(e_r^2 - 1)^{7/2}} + \frac{e'_4(3e_r^4 + 24e_r^2 + 8)}{8(e_r^2 - 1)^{9/2}} + \frac{e'_5(15e_r^4 + 40e_r^2 + 8)}{8(e_r^2 - 1)^{11/2}}, \quad (\text{C2a})$$

$$e_1 = \frac{e'_1 e_r}{(e_r^2 - 1)^{3/2}} + \frac{2e'_2 e_r}{(e_r^2 - 1)^{5/2}} + \frac{3e'_3(e_r^2 + 4)e_r}{4(e_r^2 - 1)^{7/2}} + \frac{e'_4(3e_r^2 + 4)e_r}{(e_r^2 - 1)^{9/2}} + \frac{5e'_5(e_r^4 + 12e_r^2 + 8)e_r}{8(e_r^2 - 1)^{11/2}}, \quad (\text{C2b})$$

$$e_2 = \frac{e'_2 e_r^2}{4(e_r^2 - 1)^{5/2}} + \frac{3e'_3 e_r^2}{4(e_r^2 - 1)^{7/2}} + \frac{e'_4(e_r^2 + 6)e_r^2}{4(e_r^2 - 1)^{9/2}} + \frac{5e'_5(e_r^2 + 2)e_r^2}{4(e_r^2 - 1)^{11/2}}, \quad (\text{C2c})$$

$$e_3 = \frac{e'_3 e_r^3}{12(e_r^2 - 1)^{7/2}} + \frac{e'_4 e_r^3}{3(e_r^2 - 1)^{9/2}} + \frac{5e'_5(e_r^2 + 8)e_r^3}{48(e_r^2 - 1)^{11/2}}, \quad (\text{C2d})$$

$$e_4 = \frac{e'_4 e_r^4}{32(e_r^2 - 1)^{9/2}} + \frac{5e'_5 e_r^4}{32(e_r^2 - 1)^{11/2}}, \quad (\text{C2e})$$

$$e_5 = \frac{e'_5 e_r^5}{80(e_r^2 - 1)^{11/2}}. \quad (\text{C2f})$$

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